# The Calculating Technician

David Roberts

Piano Technicians Guild
Foundation Press, 1990

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# Preface

The Piano Technicians Guild Foundation Press (PTGFP) was formally established in 1987. Its charter is the printing of books developed from articles appearing in the *Piano Technicians Journal* (*PTJ*), the official publication of the Piano Technicians Guild Inc. (PTG). In order to implement its charter, a committee of distinguished PTG members was created: former *PTJ* editor Jack Krefting, *PTJ* historian and contributor Jack Greenfield, piano designer and teacher Delwin Fandrich, and respected PTG leader (and prime mover of this undertaking) Charles Huether.

Our choice of material for this first PTGFP publication is a series of 20 articles appearing in the *PTJ* from 1979 to 1981, entitled, "The Calculating Technician" by Cleveland chapter member David Roberts. The motivation for this choice is that these articles have apparently been among the most often requested reprints in the Guild's history. They have also been the stimulus for widespread interest in piano scale evaluation and rescaling.

A great deal of effort has gone into making this first offering a reality and it is hoped that many more books will be offered in the future. The PTGFP would like to acknowledge its appreciation for the perseverance of the committee and for the dedicated work of Yvonne Ashmore and the PTG Home Office staff. Special thanks go to Barb Fandrich, who typeset the manuscript.

# About the Author

David Roberts was born in Fort Wayne, Indiana in 1939. He moved to Arizona in 1950 and graduated first in his class of 306 at Phoenix Camelback High School, where he also became a proficient oboist. Mr. Roberts went on to earn a B.S. and M.S. in physics at M.I.T. and Case Institute of Technology, and was first oboist and guest soloist in several symphonic and chamber orchestras in Boston and Cleveland during those years.

His first eight years in industry were devoted to theoretical and experimental solid-state acoustics. In 1973, he co-founded Cleveland Crystals, Inc., a manufacturer of nonlinear and electro-optic crystals for the laser industry. Now vice president of optical engineering, he has designed numerous laser optic products which have been widely accepted by the international laser fusion community and by the commercial laser marketplace.

In 1971, following the purchase of a small grand piano for himself and his (violinist) wife Edith, Mr. Roberts developed an intense interest not only in furthering his playing skills, but also in tuning and rebuilding. His association with Case physics Professor Arthur H. Benade, author and international consultant on musical acoustics, helped inspire Mr. Roberts' subsequent experimental and analytical investigation of piano physics, especially inharmonicity and its impact on tuning and scaling.

As a craftsman member of the Guild for eight years, Mr. Roberts plied the tuning, repairing and rebuilding trades and also taught classes on tuning and scaling theory at several local, state and national PTG conventions. The demands of Cleveland Crystals finally forced him to abandon these activities in 1982, but he still maintains contact with Guild members.

# 1 Introduction

The purpose of this book is not only to introduce formulas useful to the piano technician in his or her work and to show how to use them, but to explain how to calculate these formulas by hand and by electronic calculators.

Piano technology, like most professions, is becoming increasingly sophisticated as new knowledge and improved techniques develop in piano construction, repair and rebuilding. As craftsmen, we owe it to ourselves to keep abreast of these developments whenever we can. Sometimes, however, new information comes to us in language we do not understand, such as music theory, scientific terminology or mathematical descriptions.

The use of theory or scientific terminology usually presumes that the reader (or listener) has an academic background in the subjects at hand. It is, therefore, no surprise that many piano technicians learn little from reading the *Journal of the Acoustical Society of America* and other respected periodicals which have, over many years, published numerous articles pertinent to piano technology.

The mathematics for calculating inharmonicity in vibrating piano wires was published prior to 1900 (References [1] and [2]), but it was not until half a century later that piano people started to grasp a quantitative understanding of this phenomenon so basic to piano acoustics, scale design and even tuning. We can blame some of the inadequacy of stringing scales in the smaller pianos on this lack of knowledge which was available all along, but not in a language which piano people understood. Even today, few tuners or rebuilders have more than a vague understanding of piano inharmonicity as it relates to fine tuning and proper scaling.

Part of the problem for this state of affairs is that the end result of a mathematical derivation or scientific experiment is often expressed as an algebraic formula, which frightens most technicians. This is unfortunate because, in many instances, the piano technician could ignore all the complex theoretical derivations and verbal dissertations if only he or she understood how to apply the given formula. True, it has been argued that a little knowledge, i.e., only the formula itself, is a dangerous thing in the absence of general understanding. Maybe so, but even this knowledge is a start and is not likely to be "dangerous" if ordinary caution and common sense are exercised.

In this book we will show the piano technician that calculating frequencies, cents, inharmonicity, tension, elongation, etc., from algebraic formulas is really not difficult at all. In fact, anyone who can add, subtract, multiply and divide is well qualified. We will **boldface** algebraic symbols and constants which are being discussed in the text, as well as those appearing in separate, stand-alone formulas. This extra emphasis should help the reader follow the explanations of various mathematical formulations which we will be discussing throughout this book.

# 2 Scaling Formula Algebra
## (String Tension Calculations)

Anyone who is well versed in the numerical evaluation of algebraic formulas may wish to skip this chapter and go on to Chapter 3. On the other hand, we have a calculation example which I believe will be of interest to everyone. In any case, we can begin the subject of piano scaling algebra by considering a formula for string tension, represented below by the letter **T**:

$$\mathbf{T} = \left(\frac{\mathbf{PLd}}{\mathbf{K}}\right)^2 \left[1 + \mathbf{A}\left(\frac{\mathbf{D}^2}{\mathbf{d}^2} - 1\right)\right]$$

You must understand that this expression is just a shorthand notation for a series of simple arithmetic steps. If you are bothered by the idea of adding, subtracting, multiplying and dividing letters of the alphabet, just remember that we will eventually replace these letters with some real numbers. In the meantime, if you would rather think of each letter as a number, that is O.K. The idea here is to calculate the quantity on the right side of the "equals" sign. If we do this, we will have the answer we seek; namely, the pounds tension in a wound string whose speaking length is represented by the letter **L**, core wire diameter by the letter **d**, overall diameter (including winding) by **D** and pitch (frequency) by the letter **P**. The units to be used and numerical values of the constants **A** and **K** will be discussed later.

Notice that the expression to the right of the "equals" sign consists of two main parts, one being the quantity

$$\mathbf{T} = \left(\frac{\mathbf{PLd}}{\mathbf{K}}\right)^2$$

and the other being the quantity

$$\left[1 + \mathbf{A}\left(\frac{\mathbf{D}^2}{\mathbf{d}^2} - 1\right)\right]$$

The fact that these two bracketed quantities are written next to each other indicates that they are to be multiplied together once each part is calculated individually. For example, if the first part should turn out to be 144 and the second part turns out to be 2.5, the final answer we seek would be 144 times 2.5 which is 360: i.e., the tension **T** would be 360 pounds. Now let us calculate the quantity

$$\left(\frac{\mathbf{PLd}}{\mathbf{K}}\right)^2$$

First, notice the number 2 to the right and slightly above the expression in parentheses. This is a shorthand notation telling us to "square" the entire quantity within the parentheses, which means to multiply the quantity by itself. Before we actually do this squaring operation, we will first have to figure out

$$\left(\frac{\mathbf{PLd}}{\mathbf{K}}\right)$$

To do this, note that when two or more letters are placed together on the same line, such as **PLd**, this means the successive multiplication steps **P** times **L** times **d**. For example, if **P** is 2, **L** is 6 and **d** is 4, then **PLd** is just 2 times 6 (which is 12) multiplied by 4 (which gives a total of 48). It does not matter in what order you do these multiplication steps: if you want to multiply **P** times **d** first to get 8 and then multiply by **L** to get 48, that is all right, too.

Next, note that when one or more letters appear *over* one or more different letters, such as we have with

$$\frac{\mathbf{PLd}}{\mathbf{K}}$$

(sometimes written **PLd/K** to save space), this means that you take the number on top and divide it by the number on the bottom. For instance, we have already determined that **PLd** is 48 in the present example, so 48 is the "number on top." If the letter **K** turns out to be 4, then we should divide 48 by 4, which equals 12. Now that we have determined that **PLd/K** has a numerical value of 12, we can finally "square" it, as described above, so we have 12 times 12 which is 144. To repeat, $(\mathbf{PLd/K})^2$ is 144.

Here is a test to see what you have learned so far: suppose **P** = 3, **L** = 6, **d** = 2 and **K** = 4. Can you calculate $(\mathbf{PLd/K})^2$? For the answer, see below.*

Now we want to describe how to calculate the expression in square brackets, namely

$$\left[1 + \mathbf{A}\left(\frac{\mathbf{D}^2}{\mathbf{d}^2} - 1\right)\right]$$

We remarked earlier that once we figure out the individual quantities

$$\left(\frac{\mathbf{PLd}}{\mathbf{K}}\right)^2 \quad \text{and} \quad \left[1 + \mathbf{A}\left(\frac{\mathbf{D}^2}{\mathbf{d}^2} - 1\right)\right]$$

we simply multiply these two quantities together to get the pounds of tension in the string.

**(Answer: nine squared, which is eighty-one.)*

There are different ways to approach the calculation of the quantity in square brackets. This may depend on personal preference or on whether you have a calculator to help you or even on what kind of calculator you might have. For now, let us first figure out the quantity in parentheses, i.e.,

$$\left(\frac{D^2}{d^2} - 1\right)$$

then multiply this result by the constant **A** and finally add 1. In order to calculate the quantity in parentheses, recall the rules for "squaring" and also for dividing one number by another number written under it. For example, suppose $D = 8$; then $D^2$ is 8 times 8 which is 64. Likewise, if $d = 4$, then $d^2$ is 4 times 4 which is 16. Therefore, the ratio of $D^2$ to $d^2$ (sometimes written $D^2/d^2$ or $(D/d)^2$ in order to save space) is just 64 divided by 16 which is 4. Since the parentheses surround both $D^2/d^2$ and the number 1, this implies that we must first subtract 1 (as indicated) from $D^2/d^2$ before we can multiply by **A**. For example, if **A** is 1/2, then

$$A\left(\frac{D^2}{d^2} - 1\right)$$

is just 1/2 times (4 – 1) or 1/2 times 3, which is 1.5. Finally, we have to add this result to the number 1 (as indicated) to complete the quantity in square brackets, so [1 + 1.5] = 2.5. Now we can multiply this result for the value of the quantity in square brackets by the quantity $(PLd/K)^2$, which was calculated earlier to be 144. Therefore, 2.5 times 144 is 360; i.e, the string tension in this example is 360 pounds.

Now that we have done a sample calculation using simple numbers, let us do a more realistic calculation using real (rather than make-believe) numbers. In the real world, the constant **K** in our tension formula has a value **20833**. Also, the constant **A** in the real world depends on what material is used to wrap the wound string; if the wrap is copper, then **A** has the value **0.89**; if it is iron, then **A** is **0.79**; and if the wrap is aluminum, then **A** is **0.27**. In order to calculate the tension in pounds with the formula we have given, it is necessary that we express the quantities **P**, **L**, **d** and **D** in the proper units. In this case, we should express speaking length **L** in inches and pitch **P** in cycles-per-second (sometimes abbreviated Hertz or simply Hz); core diameter **d** and overall diameter **D** should be expressed in "mils," which is shorthand jargon for "thousandths-of-an-inch." Just for fun, let us calculate the string tension for the lowest **F (F1)** in a certain Bechstein concert grand: this copper wound monochord has **L** = 75 inches, **P** = 43.7 Hz., **d** = 63 mils and **D** = 145 mils. Therefore, the tension is

$$\begin{aligned} T &= \left(\frac{43.7 \times 75 \times 63}{20833}\right)^2 \left[1 + 0.89\left(\frac{145^2}{63^2} - 1\right)\right] \\ &= \left(\frac{206483}{20833}\right)^2 \left[1 + 0.89(5.3 - 1)\right] \\ &= (9.91)^2\left[1 + 0.89(5.3 - 1)\right] \\ &= 98.2 \times 4.83 \\ &= 474 \text{ pounds} \end{aligned}$$

This is one of the highest string tensions I have come across in a piano scale. The lowest string tension you are likely to find in a modern piano (around 100 pounds) is usually at the bottom end of the treble bridge in small pianos with no wound treble unisons. You can calculate the tension in a plain string by noting that the quantity in the square brackets is just 1 for a plain string because **D = d** when there is no wrap. Therefore, if the example Bechstein string had *no* wrap on it, the tension would be just 98.2 times 1 or 98.2 pounds in order to sound at the correct pitch, **F1**.

For an alternate and even easier to use tension formula, refer to Appendix 2.

# 3 Essentials of Good Scale Design

So far, we have concentrated on the step-by-step solution of a typical algebraic formula which might occasionally confront the piano technician. Our example was the calculation of tension in a plain or wound piano string. Hopefully, anyone who can add, subtract, multiply and divide was able to follow this chapter...at least, that was the objective.

Explanations of how to make this calculating even easier by using various kinds of electronic calculators can be found in Appendices 1 through 3.

We started with string tension because most technicians are comfortable with the idea of tension, even if they do not know how to calculate it. Three of the most frequent occasions where a piano technician can utilize his mathematical tools involves replacement of missing strings, scale evaluation and scale modification. In these cases, tension is certainly one factor to consider, but there are several important acoustical factors which should also be considered. Therefore, we will concentrate on formulas pertinent to piano scale evaluation and modification for the next few chapters. We will describe the step-by-step solution for each of these formulas and explain how to use them in practice. First let us discuss some aspects of piano scales without resorting to formulas.

As many of you are aware, one of the most common problem areas in piano scales is the transition region from treble to bass, particularly in pianos smaller than about seven feet. For instance, you have probably encountered one or more of the following symptoms which are characteristic of inadequate scale design in this transition region:

- You cannot set a good temperament, particularly if wound strings are present.
- You cannot tune smoothly in the various tuning test intervals simultaneously.
- You cannot tune notes in the upper treble to be in tune with the transition region.
- Some hammers need frequent or excessive voicing.
- No amount of voicing gives a good aural transition from plain to wound strings or from treble to bass.
- Lower (plain wire) treble notes have unstable tuning relative to nearby notes on the scale.

Although the stringing scale is not always to blame for some of these problems, there is a good chance the scaling is at fault if several of these symptoms are present together. In this case, the scale can be improved considerably by examining three important acoustical quantities for each unison in and adjacent to the suspect portion of the scale. These acoustic quantities, which we will eventually learn to calculate, are as follows:

- string inharmonicity
- unison loudness/sustaining factor
- hammer/string contact time

The general idea in good piano scale design is that all three of the above acoustic quantities should change in a *smooth* and proper fashion from one end of the keyboard to the other, including the tricky transition region from plain to wound strings. At the same time, each individual string tension should preferably be maintained below a conservative upper limit which, for pianos of basically modern design, is about 60% of the breaking tension. A formula for this limiting tension is:

$$T_{max} = 0.557d^{1.667}.$$

In this formula, $T_{max}$ represents the maximum safe tension in pounds and $d$ represents the (steel) wire diameter in mils. As an example, the core wire diameter for the Bechstein F1 monochord is $d$ = 63 mils, so 63 raised to the power 1.667 is approximately 999 (use the $y^x$ button on your calculator as described in Appendix 2). Therefore, the so-called safe upper limit for this string is $T_{max}$ = 0.557 times 999, which is 556 pounds. Recall from Chapter 2 that the actual string tension is 474 pounds, which easily falls within the guideline mentioned above.

I want to emphasize that mere conformance to the "safe tension" guideline is not a proper way to design wound or plain strings in a piano scale. It is a desirable condition, but one should principally consider the three acoustic quantities mentioned above in order to replace a group of missing strings or arrive at a proper scale modification. In addition to the aural clues for detecting problem scales, you can often spot a problem scale just by looking at the piano. For instance, smoothness in the unison-to-unison variation of the three acoustical quantities implies that half-sizes of piano wire should not be skipped in the treble stringing scale. Although you may approach a point of diminishing return when you get to the larger wire gauges, inclusion of all half-sizes during restringing or scale modification can only improve the scale, never degrade it.

Another example of a potential scaling problem is a treble bridge that does not make sufficient "doglegs" under the treble plate struts in order to maintain a smoothly accelerating increase in speaking lengths from C88 on down. If this happens, there is not much you, the technician, can do about it. No matter how much you jockey wire sizes in such regions of the scale, you will never be able to get all three of the acoustical quantities to vary in a smooth fashion. You can achieve a compromise, but you may have to voice some hammers severely in the process.

Perhaps the most troublesome scaling problems are at the lower end of the treble bridge, particularly in pianos in which this portion of the bridge reverses its curvature and exhibits a significant hook back toward the hammer line. This causes the speaking lengths to fall far short of their proper scaling values, resulting in tuning instability, stridency or tinniness and/or loss of power, As this book proceeds, we will show how this situation can be improved considerably by the addition of properly designed wound strings to the treble bridge. This approach can be quite successful if the hook is relatively strong, but does not begin until fairly close to the bass end. If the hook starts in the middle of the treble bridge and has a slow sweeping backward "S" shape, then hopes for improving the scale are diminished.

# 4 Hammer/String Contact Time

We will now concentrate on formulas for use in piano scale evaluation and modifications. The formulas for calculating tension of piano strings (Chapter 2) and also the formula for calculating the approximate safe upper limit for string tension (Chapter 3) have already been presented.

We have pointed out, however, that tension considerations alone are insufficient to evaluate or modify a piano scale or to design a sizeable number of missing strings.

If a piano is missing a few strings, you may be able to determine what they were from the hitch-pin layout and measurements of adjacent wire sizes. If this is not enough, there is a "rule of thumb" attributed to William Braid White for treble scaling in pianos of basically modern design (Reference [3]). This rule states that treble unisons should start with 13 or 13-1/2 music gauge at C88 and increase by half-sizes every five unisons; at the same time, the speaking lengths should increase by approximately 5-2/3% per unison starting with about 2" at C88. Braid White's rule is seldom followed to the letter, but rarely do piano manufacturers stray far from this design precept. Most pianos, regardless of size or sequence of wire gauges, still *average* about five unisons per half-size down to about Middle C. At Middle C the wire gauge is usually 17-1/2 if 13 gauge was used at C88, or else 18 gauge if 13-1/2 was used at C88. The latter is common in concert grands and so-called high tension scales.

But now we come to the question of what to do below Middle C where speaking lengths often do not continue to increase at the rate specified above (problems due to foreshortening of proper scaling lengths were discussed in Chapter 3). Also, what do we do about the design of wound strings on the bass bridge? How do you blend plain and wound strings?

Although "rules of thumb" may occasionally give sufficient clues to cope with the missing strings problem, it is clear that more rigorous rules are desirable, especially if you wish to evaluate or modify a section of the scale. In Chapter 3, we described a number of aural and visual clues to possible scaling problems. It was indicated that one can usually resolve the question of faulty scaling by calculating three important acoustical quantities for each unison in the suspect part(s) of the scale:

- string inharmonicity
- unison loudness/sustaining factor
- hammer/string contact time

Measurements show that, with few exceptions, the above acoustical quantities tend to change from note to note in a remarkably *smooth* fashion in good scales. As this book proceeds, we will describe in some detail what effect each of these three quantities has on such important attributes of a piano as its tunability, tone and voicing uniformity. Then it will become evident why these quantities *should* change smoothly in a good scale.

Let us deal with the last acoustical quantity first; namely, hammer/string contact time—i.e., the period of time during which at least some portion of the hammer felt is under momentary compression due to hammer contact with the strings. Most piano technicians know that a piano tone contains a number of higher partials (overtones) in addition to the 1st partial (fundamental). To a great extent, piano tone quality is determined by the relative strengths of these partials. There are many factors which determine the relative partial strengths by the time a piano tone finally reaches our perceptual senses, but the one we are particularly concerned with in regard to piano scale design is the hammer/string contact time. Although a large number of partial tones are excited by the hammer striking the string, some of them are damped out due to lingering contact of the hammer felt with the string(s). Partials whose period of vibration is less than the hammer/string contact time contribute very little to the piano tone. Without going into a great deal of complicated physics, suffice it to say that the hammer/string contact time will change in a smooth fashion from note to note if the mass, shape and softness of the hammers change smoothly and if a certain ratio containing measurable or calculable quantities also changes smoothly (Reference [10]). This ratio can be written algebraically as **NT/H** and can be calculated for each unison of interest as follows. First, calculate the "unison tension," **N** times **T**, where **N** is the number of strings in the unison and **T** is the tension in each string. Then divide this unison tension by the strike point distance **H**, which is the distance from the capo bar or agraffe to the point where the hammer touches the string(s).

For example, suppose we have a trichord unison

(**N** = 3) with string tension **T** = 160 pounds and strike point distance **H** = 6 inches. Then the ratio **NT/H** will be 3 times 160 (which is 480) divided by 6 which equals 80. The physical significance of this number is that the larger it is (everything else being equal), the faster the hammer will rebound from the strings, resulting in a larger percentage of upper partials. Since the strike point distance is quite small at C88 and much larger at A1, whereas unison tension changes much less over the scale, we expect the ratio **NT/H** to *increase* smoothly from note to note as we proceed *up* the keyboard in a well-scaled piano. Though you might therefore conclude that the higher keyboard notes have a larger percentage of upper partials, this is not the case because several other factors act to decrease this percentage.

Let us look at how **NT/H** changes in a Steinway concert grand near the bass/treble break. In the following table, **m** is the number of the note as it lies on the keyboard and the other letter symbols were explained previously. The interesting feature of this scale is that, although there is a 32% jump in strike point distance **H** from unison 20 to unison 21 (reflecting a similar 32% jump in speaking length), there is a remarkably *smooth* transition in the hammer/string contact time factor **NT/H** across this break due to a corresponding jump in unison tension. This is as it should be, since it helps ensure that there will be only a very small difference in the corresponding relative partial strengths for these two notes, hence minimizing voicing problems across this transition.

| m | N | T | H | NT/H |
|---|---|---|---|---|
| 18 | 3 | 160 | 7.4 | 65 |
| 19 | 3 | 152 | 7.1 | 64 |
| 20 | 3 | 151 | 6.8 | 67 |
| bass/treble break | | | | |
| 21 | 3 | 201 | 9.0 | 67 |
| 22 | 3 | 207 | 8.6 | 72 |
| 23 | 3 | 208 | 8.2 | 76 |
| 24 | 3 | 213 | 7.8 | 82 |

Let us look at another example, this time a much smaller grand whose scale was modified several years ago. Note the unusually large jump in **NT/H** in the original scale and how much more smoothly it changes from unison 24 to unison 31 in the modified scale.

| | NT/H | |
|---|---|---|
| m | original scale | modified scale |
| 24 | 83 | 83 |
| 25 | 117 | 78 |
| 26 | 126 | 82 |
| bass/treble break | | |
| 27 | 64 | 82 |
| 28 | 70 | 81 |
| 29 | 78 | 85 |
| 29 | 78 | 85 |
| 30 | 88 | 87 |
| 31 | 89 | 89 |

I want to emphasize at this point that, just as tension considerations alone are not sufficient to evaluate or modify a piano scale, neither is consideration of only the hammer/string contact time factor. We still have to deal properly with string inharmonicity and the unison loudness/sustaining factor. Interestingly, we will see that consideration of **all** three of these acoustical quantities will sometimes lead to conflict concerning the direction in which to proceed when modifying a scale, so compromise must be used. More about this later.

You may be wondering how scaling errors of the sort indicated in the small grand above could have survived listening tests or arisen on the drawing board in the first place. One reason is that it is doubtful that piano manufacturers approached scale design from a physical acoustics point of view, such as suggested here. As far as I know, mathematics used near the turn of the century, and for many years thereafter, was confined primarily to the layout of bridges, agraffe (or capo bar) lines and strike points. These layouts were done according to rather simple, empirically determined, geometric relationships (References [3] and [4]). Thus, the acoustical quantity **NT/H** was probably never considered at all. The rough scaling in the small grand mentioned above quite possibly may have resulted from compromising with other problems. For instance, it was mentioned in the last chapter that foreshortening of the proper scaling lengths in the lower treble due to a hook in the bass end of the treble bridge can cause increased stridency in these notes. This was, in fact, the situation with the aforementioned small grand. Rather than design wound strings on the bass end of the treble bridge, which is a proper way to cope with this problem, the manufacturer apparently chose to add a third string to the two uppermost wound bichord unisons on the bass bridge. The resulting increase in loudness coupled with a higher percentage of upper partials therefore made these two unisons blend better aurally with the strident lower treble unisons. Although trade-offs of this sort work to a limited extent, usually the problems simply become compounded.

Piano manufacturers today are more knowledgeable in physical acoustics. As a result, scaling in some of the newer small grands and verticals has improved significantly. It is hoped that piano rebuilders who wish to rectify scaling problems in many otherwise fine older instruments will utilize the same knowledge of acoustics being used by the more progressive manufacturers of today.

# 5 Unison Loudness/ Sustaining Factor

We have pointed out that one can usually resolve the question of suspected flaws in a piano scale by calculating three acoustical quantities for each unison in the suspect part(s) of the scale. In approximate order of importance, these quantities are as follows:

- string inharmonicity
- unison loudness/sustaining factor
- hammer/string contact time

Our rule for good scales is that each of these acoustical quantities ideally should change in a smooth and proper fashion from unison to unison. We will discuss exceptions as we go along. At the same time, individual string tensions should be maintained below a safe limit, as discussed in Chapter 3. The order of priorities above helps us decide what to do in those instances when it is impossible to get all three acoustical quantities to change in a perfectly smooth fashion simultaneously. More about this also as we go along.

In the previous chapter, we discussed hammer/string contact time and its relation to tone production. As it turned out, we did not find it necessary to calculate this contact time *per se*. Instead, we demonstrated how to calculate a simple ratio of measurable and calculable quantities which is closely related to hammer/string contact time, and stated that optimum smoothness in voicing requires that this ratio increase **smoothly** as we proceed up the keyboard. This ratio was given as **NT/H**, where **T** represents the individual string tensions in a unison, **N** is the number of strings in the unison and **H** is the so-called strike point distance. In most scales, the quantity **NT/H** indeed changes quite smoothly in the treble trichord sections, because the unison tension **NT** changes only slightly (usually decreases towards the treble end) while the strike point distance **H** changes in approximate proportion to the speaking lengths. We also saw in the previous calculations of **NT/H** in a Steinway concert grand, that this ratio in a good scale can maintain a remarkable smoothness even across the bass/treble break, in spite of sudden large jumps in **NT** and **H** individually at this transition. Beyond this, one usually finds that the ratio **NT/H** does not always change as smoothly as one would like (for instance, in the wound monochord to bichord transition), even in the best scales. This is apparently in deference to maintaining a smooth change from unison to unison in the other two acoustical quantities above, as we shall see.

Now let us discuss the second of our three acoustical quantities, the so-called unison loudness/sustaining factor. The physical significance of this factor is that the larger it is (everything else being equal), the more quickly the vibratory energy in the unison is transferred to the soundboard, thus producing a louder but less sustaining tone. This factor is related to something physicists call acoustic wave impedance. The theoretical background for wave impedance and for perceived loudness in piano tones is beyond the scope of this book, but can be found in Reference [10], particularly Chapters 13 and 17.

An algebraic expression for the loudness/ sustaining factor, which we shall henceforth denote bythe letter **Z**, is as follows:

$$Z = N^{a} d \sqrt{T\left[1 + A\left(\frac{D^2}{d^2} - 1\right)\right]}$$

All of the letter symbols in this formula, except the exponent **a**, have been used and defined in previous chapters: **T** is the tension (in pounds) of the individual strings in the unison, **N** is the number of strings in the unison, **d** and **D** are the steel wire diameter and overall diameter (in mils), and the number **A** is **0.89, 0.79, 0.27** or **0**, respectively, depending on whether the strings are wrapped with copper, iron, aluminum or are not wrapped at all. This formula for **Z** is an updated version of the one in the handout sheets for my convention classes the past few years, but it is written in a different form. The principal difference is in the power (or exponent) of **N** which I have denoted by the letter **a**. More about the numerical value of **a** in a moment.

As an example of how both **Z** and the hammer/ string contact time factor **NT/H** change across several scaling breaks in a good instrument, let us look at unisons **m** = 8 through 21 in a (1923) Steinway concert grand. (See table at top of next page.)

Here, we have a break from copper wound mono-chords to iron wound bichords from **m** = 8 to **m** = 9, then a break from iron bichords to iron trichords (**m** = 13 to **m** = 14) and, finally, a break from iron trichords to plain trichords which also happens to coincide with the break from bass bridge to treble bridge (**m** = 20 to **m** = 21). An examination of the table reveals that, despite the wide range of unison types represented here and despite the

| m | N | T | Z | NT/H |
|---|---|---|---|---|
| 8 | 1 | 330 | 2202 | 35 |
| 9 | 2 | 243 | 2051 | 52 |
| 10 | 2 | 254 | 2059 | 55 |
| 11 | 2 | 249 | 1942 | 55 |
| 12 | 2 | 238 | 1791 | 54 |
| 13 | 2 | 211 | 1531 | 49 |
| 14 | 3 | 193 | 1603 | 69 |
| 15 | 3 | 182 | 1464 | 67 |
| 16 | 3 | 177 | 1388 | 67 |
| 17 | 3 | 177 | 1352 | 69 |
| 18 | 3 | 160 | 1202 | 65 |
| 19 | 3 | 152 | 1124 | 64 |
| 20 | 3 | 151 | 1100 | 67 |
| 21 | 3 | 201 | 1034 | 67 |

wide range of individual string tensions **T**, the calculated values of the loudness/sustaining factor **Z** undergo a remarkably *smooth decrease* from unisons 8 through 21. There is a small reversal in this trend at **m** = 9 to **m** = 10 and from **m** = 13 to **m** = 14, but these are rather minor. The table also reveals that the factor **NT/H** *increases* from unisons 8 through 21, as we predicted in previous discussions, but the change is somewhat rough across the various scaling breaks with the exception of the bass/treble break. Again, this is in conformance with our earlier assertion that smoothness in **Z** usually has priority over smoothness in **NT/H** when it is impossible to get both to change smoothly simultaneously. What is even more remarkable about this scale is that the calculated string inharmonicities also change smoothly through all these scaling breaks, but I will defer discussion of inharmonicity until the next chapter.

A couple of clarifying statements are in order at this point. The first concerns the numerical value of the exponent **a** in the formula for **Z**. I used **a = 0.4** in the **Z** calculations for the Steinway grand because this gives the smoothest change in **Z** across all the scaling breaks. If I do the same set of calculations on the Bechstein concert grand that we have referred to earlier, then I find that **a** = **0.6** gives the smoothest change in **Z** for this instrument. These results are quite similar, so my recommendation is to average them and henceforth use **a** = **0.5** in the formula for **Z**. Theoretically, the precise value of **a** depends on how closely matched, physically, the unison strings are, how well they are tuned during use and other factors. Rather than rely strictly on theory to tell us what a good value of **a** should be, I have chosen to let two fine concert instruments tell us, as I have just described. Incidentally, if we let **a** = **0.5**, then **N** to the power **0.5** is the same thing as the square root of **N**, so we can rewrite our formula for **Z** in even simpler form as

$$Z = d\sqrt{NT\left[1 + A\left(\frac{D^2}{d^2} - 1\right)\right]}$$

As an example, consider our old friend the Bechstein F1 monochord: **d** = 63 mils, **N** = 1, **T** = 474 pounds and the value of the quantity in square brackets is 4.83 (see Chapter 2). To calculate **Z**, we multiply **N** times **T** times 4.83, which is 1 x 474 x 4.83 = 2289.4. Then take the square root (use the square root button on your calculator), which gives 47.85. Finally, multiply this by **d** to get 47.85 x 63 = 3015; i.e., **Z** = 3015 for this unison. Do not bother figuring out what the units are for this number. It is not important. The important thing is that, with an exception to be described in a moment, these calculated values of **Z** should *decrease smoothly* from the bass through treble sections of the piano, hopefully in a fashion similar to that of the Steinway scale just described. It is not necessary, however, that every piano have the same values of **Z** at the same note positions.

The second clarifying statement I wish to make is that you will occasionally encounter a piano where **Z** takes a sizeable jump at the bass/treble break which may or may not be legitimate from a modern scaling point of view. Our rule that **Z** should decrease smoothly as you proceed up the scale presumes that the bridge/soundboard/rib structure has a smooth response to string excitation along the entire length of the scale. If, as sometimes happens, the manufacturer has made the geometry and placement of the bridges so that the bass bridge/soundboard/rib structure is much stiffer and/or more massive and therefore less responsive than its treble counterpart, then a smooth loudness/sustaining transition *requires* an offsetting jump *downward* in **Z** when making the transition to the treble bridge. If however, the two bridges have comparable response to mechanical excitation at this break, then such a jump in **Z** is probably not legitimate and most likely was made to compensate other scaling errors. The most common example of this is making the topmost bass notes extra loud to compensate aurally for stridency (high inharmonicity) in foreshortened, unwound lower treble notes, especially in the smaller pianos. One tipoff, then, could be to calculate inharmonicities on either side of this break to see whether they change smoothly or take a large jump upward when making this transition to the treble bridge. If the latter, you may be justified in eliminating at least part of the jump in **Z** when rescaling for smoother inharmonicities. Uncertain situations like this admittedly make scale evaluation or modification difficult at times. For this reason, it is a good idea to test any modification near the break before unstringing the piano and preferably after hammers have been reconditioned or replaced and the action is in good regulation.

In subsequent chapters, we will show more graphically how to use the loudness/sustaining factor **Z** during scale evaluation or modification, but first we need to discuss what is probably the most important acoustic quantity of all...string inharmonicity.

# 6 Inharmonicity Calculations

In Chapter 5, we indicated that one can resolve the question of suspected flaws in a piano scale by calculating three acoustical quantities for each unison in the suspect part(s) of the scale. In approximate order of importance, these quantities are:

- string inharmonicity
- loudness/sustaining factor
- hammer/string contact time factor

Our rule for good scales is that, with the possible exception of the loudness factor at the bass/treble break, each of these acoustical quantities ideally should change in a smooth and proper fashion from unison to unison across the entire keyboard. We have already discussed the 2nd and 3rd quantities in Chapters 4 and 5 and also the role which string tension plays in scale evaluation. In this chapter, let us tackle the most important acoustical quantity of all, string inharmonicity. This quantity is vitally important because it determines the tunability of a piano scale and is an important factor affecting the voicing of piano tones. No amount of voicing of the hammer felt or regulation of the action can affect string inharmonicity in any way. Hence, once the piano has been strung, one is stuck with whatever scale flaws may exist with respect to string inharmonicity, unless someone is willing and able to modify that scale.

Actually, there are several sources of inharmonicity in pianos. These include wire stiffness, non-rigid string terminations, soundboard resonances and wire non-uniformities, such as caused by corrosion, over-stretching, uneven wrapping of bass strings or variations in wire diameter during manufacture, the short, bare segments at the ends of wound strings, and improperly designed swaged regions near the wrapped ends. Even accumulated debris or corrosion between adjacent wrap turns can cause inharmonicity. We can rectify some of these problems. Soundboard resonances and motion of the bridges cannot be eliminated. Except for these last two effects, the two most prominent sources of inharmonicity in a well constructed instrument are:

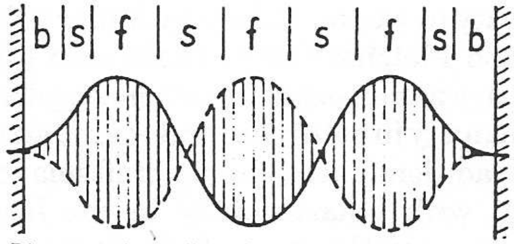
- inherent stiffness in the piano wire
- presence of unwrapped ends on wound strings

The (predictable) inharmonicity caused by these sources, unlike the other forms of inharmonicity just mentioned (unpredictable and random), are not necessarily problematic and can even have some musical virtue if dealt with properly by both the scale designer and the tuner.

The piano industry as a whole apparently did not consciously grasp the concept of string inharmonicity, let alone deal with it on a quantitative basis, until more than a half century after the effect was first treated mathematically (References [1] and [2]). Around 1938, Railsback measured note frequencies on several grand and upright pianos and demonstrated that tuners "stretch" octaves (Reference [5]). This phenomenon was correlated with string inharmonicity in a paper by Schuck and Young in 1943 (Reference [7]). However, it was not until contributions were made to the subject of inharmonicity in wound strings by Miller (1949) (Reference [8]) and Fletcher (1964) (Reference [9]) that the piano industry had an explicit mathematical formulation for inharmonicity in both plain and wound strings. Interestingly, traditional (empirical) craftsmanship coped rather well with inharmonicity in the large concert grands, even near the turn of the century. However, without adequate knowledge of acoustics, these same makers of excellent, full-sized instruments did not necessarily cope properly with string inharmonicity as the smaller instruments were developed. This probably reflects the fact that inharmonicity is much easier to cope with in the large instruments and not as much painstaking development was given to the smaller pianos.

Today, the piano industry is more knowledgeable about inharmonicity, but I believe that manufacturers have kept this information pretty much to themselves. Although I do not have the resources of a modern piano manufacturer, I have been able to determine a reasonably accurate algebraic formula for the inharmonicity in piano strings, taking into account both wire stiffness and also unwrapped ends on wound strings. This formula was based originally on the theoretical works of both Miller and Fletcher, but I have since modified the formula to agree more closely with a number of wound string experimental data obtained by myself and by Lou Day of the Denver Chapter of the Piano Technicians Guild (Reference [11]).

Before presenting this formula, let us first refer to the following sketch and discuss inharmonicity effects produced by stiffness and windings.



Piano string vibrating at the 3rd partial.

Here, we represent a vibrating string at an instant in time when its moving segments have reached their maximum amplitude (solid line). At 1/2 period of vibration later, these same segments would coincide with the dotted line and at other times would lie somewhere in between. The termination conditions in a piano force the string to bend near its ends in the regions marked **b** (Reference [6]). I have marked the other sections of this string by the letters **s** and **f** to indicate that these sections, for the most part, either remain "straight" or are "flexing" as the string vibrates. Theoretically, any stiffness in the wire in the flexing regions will cause them to flex less easily, thus raising the frequency of vibration. This is because a stiffer object always has higher natural vibration frequencies, everything else being equal. The more flexing regions there are, the greater this effect will be, which explains why the higher partials have greater inharmonicity. If wrap is added to the string, theory tells us that two things will happen. First, the wrap in the flexing regions will *increase* inharmonicity because the wrap adds inertia plus its own stiffness which further inhibits the flexing motion. Secondly, however, adding wrap also increases the inertia to transverse motion, which slows down the vibration frequency. Thus, the string must be pulled up in tension to maintain the string at the same pitch. The higher tension partially overcomes the resistance to flexing in a stiff wire and more than offsets any flexural inertia or additional stiffness due to the presence of the wrap itself, particularly if the wrap turns are not touching one another. Unless debris or corrosion gets between the wrap turns, in effect causing them to be connected together, the turns will indeed be separated, due to the act of pulling the string up to pitch. In any case, the net result of adding wrap to regions **s** and **f** is to *decrease* inharmonicity. The principal difference between **b** and **f** regions is that there is very little transverse motion in the **b** regions, just flexing. Hence, adding wrap here increases inharmonicity without the offsetting effect of higher tension. This is one argument for keeping winding wraps from coming too close to the bridge and agraffe terminations. We will have more to say later about winding lengths and also a complication introduced by swaging (flattening) the core wire at the wrap ends.

With these qualitative remarks behind us, let me now present a formula for calculating inharmonicity (in cents) in plain or wound strings which takes into account the effects described above:

$$I_n = 1731\,(n^2-1)\left\{S\left(1+\frac{B}{8}\right)+\frac{3B}{1+B}\left[\left|\frac{a}{L}-\sqrt{S}\right|^3+\left|\frac{b}{L}-\sqrt{S}\right|^3\right]\right\}$$

The symbol $I_n$ is a shorthand notation for saying "the inharmonicity of the **n**th partial"; for instance $I_4$ denotes the inharmonicity of the partial **n** = 4. The symbols **a** and **b** represent the lengths (in inches) of the short, unwrapped portions of a wound string at the agraffe and bridge ends respectively. The total speaking length **L** should also be expressed in inches. The letter **S** denotes a quantity which is closely related to the mathematical "stiffness" of the steel wire and can be calculated from string tension **T** (in pounds), speaking length **L** (in inches) and steel wire diameter **d** (in mils) as follows:

$$S = \frac{d^4}{139430\, L^2\, T}$$

Finally, the letter **B** denotes a quantity that we have calculated before, namely, the weighting factor due to wrap on a wound string:

$$B = A\left(\frac{D^2}{d^2} - 1\right)$$

where **d** and **D** are the steel wire diameter and overall diameter respectively, and **A** is **0.89**, **0.79**, **0.27** or **0**, depending on whether the string is wrapped with copper, iron, aluminum or is not wrapped at all.

Let us do an example calculation. Again, let us use the familiar Bechstein F1 monochord since, for this unison, we have already shown how to calculate tension **T** and the quantity **(1 + B)**. Recall from Chapter 2 that **T** = 474 pounds, **(1 +B)** = 4.83, **d** = 63 mils and **L** = 75 inches. Before we start calculating, let me explain some features of the inharmonicity formula and also outline a plan of attack.

First, notice that we not only have the usual parentheses ( ), square brackets [ ] and curly brackets { }, but also some vertical brackets | |. The latter have a very special meaning, which is to regard the calculated expression inside these brackets as being a positive number, whether the calculation works out this way or not. Programmable calculators usually have an ABS button or a |x| button to perform this operation.

Bech F1 L=75" Diam .063
Freq 43.7 Dlow .145



Next, our plan of attack: (1) calculate **S (1 + B/8)**; (2) calculate **3B/(1 + B)**; (3) calculate the two expressions associated with the vertical brackets; (4) add these two expressions together, as indicated; (5) multiply by the result from step 2; (6) add all this to the result from step 1; (7) multiply this (i.e., the expression inside the curly brackets) by ($n^2 - 1$); (8) finally, multiply again by 1731. O.K.?

In order to calculate **S** in step 1 you must raise **d** to the 4th power; i.e., multiply **d** by itself 3 times. Thus,

$$\mathbf{S} = \frac{63 \times 63 \times 63 \times 63}{136430 \times 75^2 \times 474} = 0.0000424$$

Since **(1 + B/8)** = 1.48, then **S (1 + B/8)** = 0.0000424 x 1.48 = 0.000063, so we have completed step 1. For step 2, since **(1 + B)** = 4.83, then **B** must be 3.83, so **3B/(1 + B)** = 3 x 3.83/4.83 = 2.38. For step 3, let us assume **a** and **b** are both 0.8". Also, we know **S** now, so $\sqrt{\mathbf{S}} = \sqrt{0.000042} =$ 0.0065 (use the square root button on your calculator). Therefore, each expression in vertical brackets is (0.8/75) – 0.0065 = 0.01067 – 0.0065 = 0.0042. This is already a positive number, so the vertical brackets do not really change anything. Finally, we have to raise this number to the 3rd power, which is 0.0042 x 0.0042 x 0.0042 = 0.000000074. There are 2 terms like this, so step 4 gives 0.00000015. Step 5 says multiply this by 2.38 which gives 0.00000041. Step 6 gives 0.00006341, demonstrating that, in this particular example, the contribution to inharmonicity from the unwound ends **a** and **b** is negligible compared to the contribution from wire stiffness **S**. If we are interested in the inharmonicity of the 4th partial (**n** = 4), then step 7 gives ($4^2 - 1$) x 0.00006341 = 0.00095. Finally, step 8 gives 1731 x 0.00095 = 1.6¢.

Thus, the calculated inharmonicity in the 4th partial of the Bechstein F1 monochord is 1.6¢.

We have so far discussed nearly a dozen sources of inharmonicity, most of which are unpredictable and random in their upward or downward alterations of the various partial frequencies in a piano tone. For instance, one such source of inharmonicity is soundboard resonances. Because the ability to tune and voice a piano smoothly depends on a smooth progression of frequencies from partial to partial and from note to note along the keyboard, these random sources of inharmonicity are to be minimized or avoided whenever possible by careful design, manufacture and maintenance of the piano. The deleterious effects of soundboard resonances, unfortunately, can only be minimized effectively by increasing the size of the soundboard, which is one reason concert grands tune up more smoothly than the smaller pianos (Reference [10]).

Fortunately, the major source of piano inharmonicity, wire stiffness, is something that can be controlled by careful design, regardless of the size of the piano, although a larger piano allows more freedom of design in this respect than does a smaller piano. This source of inharmonicity is not necessarily to be minimized, only controlled in a proper fashion, for it is an integral part of the "piano sound" and has the additional virtue of partially disguising or "fuzzing out" the errors inherent in equal temperament. Only in the bass is it frequently desirable to minimize wire stiffness inharmonicity. This is because a nearly harmonic sequence of partials in bass notes will "create" in our ears the low fundamentals which soundboards (especially the smaller ones) are inefficient in amplifying. This psychoacoustic reinforcement of weak partials is sometimes referred to as "heterodyning" or "mixing" of the (stronger) partials. This phenomenon is made possible by the way in which our ears actually distort the sounds we hear (Reference [10]). Again, the concert grand is best able to take advantage of this effect because of the very low inharmonicity in its long wound strings.

There is another source of inharmonicity which is sometimes important in wound strings and that is the unwound segments present between the wrap ends and the agraffe and bridge terminations. Earlier in the chapter, we presented a formula for inharmonicity in plain and wound strings which takes into account wire stiffness and these unwound end segments, but we did not discuss the formula much beyond a sample calculation. Since I am sure some of you have questions about this formula, let us discuss where it came from and what its limitations are. First, although it may appear to be written only for wound strings, a little head scratching reveals that it simplifies to $\mathbf{I_n = 1731(n^2-1)S}$ in the case of a plain string because the weighting factor **B** is just zero when there is no wrap (i.e., when $\mathbf{D} = \mathbf{d}$).

Secondly, you may wonder why the inharmonicity is given as proportional to $(\mathbf{n}^2 - 1)$ instead of just $\mathbf{n}^2$ as I know you have seen occasionally in other references. There is no conflict here, really. What you have seen before is an expression for the inharmonicity in an idealized version of a real piano string, the actual value being virtually impossible to calculate due to the complicated nature of real piano string terminations. However, it is possible in a real string to calculate the inharmonicity in a partial *relative to* that of the fundamental ($\mathbf{n} = 1$), whatever that may be. This latter definition is the one I have used and is more meaningful to piano technicians because it nicely sidesteps the concept of inharmonicity in the fundamental. If you use an electronic tuning aid to find the number of cents by which some partial number **n** is "sharp" of its corresponding harmonic value, you measure the cents deviation for both the fundamental and the **n**th partial and then subtract one from the other to get the relative inharmonicity. Let me remind you, however, that this result may have to be corrected for the equal temperament normalization of your electronic aid; for instance, subtract 2.0¢ if **n** = 3, 6, 12, 24, etc. or add 13.7¢ if **n** = 5, 10, 20, etc. Octave partials need no correction.

Thirdly, you should know that the inharmonicity formula is a slightly doctored-up version of the theoretical formulations of Miller (Reference [8]) and Fletcher (Reference [9]).

Fletcher is essentially responsible for the first term in curly brackets { }, although I have added the "fudge factor" **(1 + B/8)** to bring my theoretical formulation more into line with measurements on wound strings performed by Lou Day (Reference [11]) and myself. This factor effectively gives an inharmonicity constant for the lower partials in the heavier bass strings which is progressively larger than predicted by Miller's formulation and seems to be supported by the measurements of Schuck and Young (Reference [7]). (Refer to Chapter 11 for more discussion on this matter.) Physically, the factor **(1 + B/8)** accounts for an additional resistance to flexing in wound strings (and hence an increased inharmonicity) due to the flexural inertia of the winding plus its own stiffness and perhaps other factors.

Miller is essentially responsible for the second expression in the curly brackets, although I introduced the subtractive terms $\sqrt{\mathbf{S}}$ to account in an approximate way for the bending that occurs in piano strings near the terminations. Fletcher alluded to this effect and in fact, calculated its approximate magnitude, but Miller ignored it. The physical significance of the second expression in the curly brackets is that there are two additional contributions to inharmonicity in wound strings which are proportional to the cube of the respective length ratios **a/L** and **b/L**. The physical consequence of the $\sqrt{\mathbf{S}}$ terms is that, once you have reduced the unwound lengths **a** and **b** to an amount $\mathbf{L}\sqrt{\mathbf{S}}$, then a further reduction in these unwound lengths will *increase* inharmonicity as described qualitatively earlier in this chapter. In other words, there is an optimum length for the unwound ends on wrapped strings which gives minimum inharmonicity. If the unwound lengths are either shorter or longer than this, the inharmonicity will increase, although for physically different reasons. Although I have indicated that this optimum length is $\mathbf{L}\sqrt{\mathbf{S}}$, I want to emphasize that this is only an approximate value because the exact theoretical solution to this problem is virtually impossible due to the complicated nature of the string termination geometry. The best way to determine this optimum length for a given string is still the good, old-fashioned trial and error method but, just for fun, let us do the approximate calculation of this length for two typical wound strings: (1) a short upper bass string in a small grand having **L** = 35", **T** = 170 pounds and **d** = 42 mils and (2) a long bass string in a concert grand having **L** = 80", **T** = 360 pounds and **d** = 63 mils. In the first case, $\mathbf{S} = (42 \times 42 \times 42 \times 42) \div (139430 \times 35 \times 35 \times 170) = 0.000107$, so $\mathbf{L}\sqrt{\mathbf{S}} = 35 \times (0.01035) = 0.36$", i.e., about 3/8". In the second case, **S** = (63 x 63 x 63 x 63) ÷ (139430 x 80 x 80 x 360) = 0.000049, so $\mathbf{L}\sqrt{\mathbf{S}}$ = 80 x (0.007) = 0.56", i.e., about 9/16". These lengths are comparable to the shortest unwound lengths that we are accustomed to seeing in pianos and hint that perhaps the heavier wound bass strings should have longer unwrapped ends than the shorter, lightly wound strings.

# 7 Improving Inharmonicity Patterns

In the previous chapter, we have distinguished the predictable and, to some extent, virtuous inharmonicity caused by stiffness in piano wire from the unpredictable, random and usually problematic inharmonicity caused by several other factors.

The formula given earlier for calculating plain or wound string inharmonicity at first appears complicated. However, if you simply attack it one step at a time as was illustrated, I think you will be convinced that it is, at worst, only tedious and not really complicated. Fortunately, inexpensive programmable electronic calculators can reduce the time to evaluate this formula to a few seconds, thus removing the tedium.

I want to continue discussing inharmonicity and wound strings. First, you may recall that our inharmonicity formula predicts an optimum length for the unwound segments between the wrap ends and the bridge and agraffe terminations. This length was shown to be roughly 3/8" to 9/16", depending on the string, and essentially coincides with the bending (but barely moving) segments near the string terminations. Unwound ends which are either longer or shorter than the optimum length unnecessarily increase inharmonicity in wound strings.

There is a situation, however, in which one might deliberately make the unwound ends longer than the so-called optimum length. This is where there are lightly wound unisons on the treble bridge.

Why?

As you may know, the purpose of wrap on a string is to add mass to the piano wire without adding substantially to its stiffness. The resulting increase in tension, which is necessary to maintain the proper pitch, partially negates the effects of wire stiffness and thus lowers inharmonicity. At the same time, loudness or power in the piano tone is increased. It is impossible to achieve both of these benefits simultaneously with plain wire if speaking lengths become increasingly foreshortened relative to their proper scaling lengths, as is typical in the smaller pianos near the lower end of the treble bridge; hence, the switch to wound unisons.

Historically, the problem with putting wound strings on the treble bridge is that even the lightest, practicable wraps were still too heavy. Thus, the transition to the first wound unison was either too loud or too low in inharmonicity for a good aural transition, not to mention the tuning problems. There are several possible remedies to this situation:

- Put the treble wound strings on a separate (tenor) bridge so the speaking lengths can be used as an additional design parameter.
- Make wound unisons bichords to reduce loudness and find lighter wraps to prevent the sudden downward jump in inharmonicity.
- Lengthen the unwound segments at the wrap

ends, i.e, longer than the so-called optimum length discussed above.

The first remedy was abandoned long ago by most manufacturers. I am not sure why, but from a scaling point of view, it seems an obvious way to cope with the scaling problems in small pianos.

The second remedy has been partially successful. Bichords helped with respect to the transition in loudness, but finding a lighter wrap was not easy. The very fine iron and copper wire gauges are fragile, possess marginal holding ability and are difficult to wrap onto a core without breaking, although it is being done today to a limited extent. Perhaps the best innovation has been the advent of very lightweight, rugged aluminum wraps which seem to have good holding ability.

The third remedy above also has limitations. Our inharmonicity formula shows that inharmonicity in wound strings can be increased somewhat by lengthening the unwrapped ends.

However, you can go only so far with this approach because of two potential problems. First, if the unwound lengths reach too close to the maximum amplitude (antinode) regions for the higher partials, then the harmonic (should I say inharmonic?) structure of the tone starts to depart from that of a plain string; that is, the inharmonicity of these partials starts to deviate from a $(n^2 - 1)$ proportionality, going instead to a lower power of $n$. It may be argued that this is no big problem and that the wound strings (especially the heavier ones) do this anyhow for other reasons (Reference [7]).

Perhaps the greater problem results when the swages (flattening of the core) near the wrap ends reach into these antinode regions. The problem here is that there is a lot of flexing of the string in these regions, so the contribution to the overall string stiffness from these regions will be different in the two transverse dimensions of the swage. The result is different inharmonicity for string motion in these two vibration directions, which can cause the upper partials to beat with themselves. This argument also applies if the swages enter the bending regions near the terminations (refer to the figure and discussion in Chapter 6 if you have trouble visualizing all of this). Thus, optimizing the winding length as described earlier not only minimizes inharmonicity but also the chance for wild strings.

However, in the event one wishes to increase the unwound ends for the purpose of increasing inharmonicity without affecting loudness near the plain-wound transition on a treble bridge, we can estimate the maximum safe length as follows. Since there are some 15 to 25 significant partials in the wound unisons generally, it turns out the (flexing) antinode regions begin a distance from each string termination equal to approximately 1.5% to 2.5% of the speaking length plus one-half the optimum unwound (bending) length discussed earlier.

Thus, the swages should ideally not extend any further than about 1-1/2" from the string terminations. If they do, the swaging should be minimal, i.e., just enough to hold the winding secure but not "smashed" so flat that it makes a large difference in the resistance to bending perpendicular to and parallel to the plane of the swage.

Most string manufacturers conform reasonably well to these design precepts, some more than others. On the other hand, I have seen rather severe swages extending more than 2-1/2" from the string terminations. I have not made a definitive experimental study of the tuning and tonal differences arising from different swage and winding lengths, and I do not want to alarm anyone unnecessarily in this regard, but I thought you should at least be aware of some of the theoretical implications of careless design in these respects.

On a related subject, I can tell you from experience that it is important for the wrap lengths on wound strings to be similar, give or take 1/8", perhaps 3/16" as an extreme difference. If the winding lengths differ by as much as one-half inch, you will hear gross differences in the (in)harmonic structure, making it impossible to tune a bichord or trichord unison.

It is not particularly critical that the ends of the windings in a two or three-string unison line up with each other (except for aesthetic reasons), as long as the winding lengths themselves are equal. You can use the inharmonicity formula to confirm these statements theoretically. For instance, a one-half inch difference in the winding lengths for the G23 treble bichord in a certain 6' grand caused the 4th partials to differ in frequency by almost two cents when the fundamentals were tuned together. This represents a beat rate of one-half per second and results in a "snarl." The higher partials, of course, beat even faster. The inharmonicity formula can be shown to be in reasonably good agreement with these observations.

On still another related subject, I know some people worry that speaking lengths for unison strings on the bass bridge are not always equal. Differences of 3/16" to 1/4" are not uncommon in those pianos which have no notching (just a bevel) on the bass bridge. The inharmonicity formula will confirm that the difference in (in)harmonic structure of the unison strings in this case is very small, amounting only to a few one-hundredths of a cent at the fourth partial in a typical small grand.

Thus, any problems you may have tuning such unisons is not likely due to the lack of notching on the bass bridge. This should be no surprise really, because you see this situation even on high quality grands. Equal length unison strings are important in the middle and upper treble scale, however, because inharmonicity is 10 to 100 times greater at a given partial level and more sensitive to length variations.

# 8 Wound Strings

## (Design/Order/Install)

Let us continue our discussion of wraps for wound strings in more detail. Presently, there are three basic types available from the half dozen or so stringmakers in this country and Canada. These are solid copper, iron and aluminum. The iron is usually copper dipped or electroplated to prevent corrosion, although you have no doubt encountered older pianos with (dull) bare iron wraps and possibly even red brass and other materials. Currently available copper wrap gauges along with their diameters in mils (thousandths of an inch) are given in Table 8.1.

TABLE 8.1 W/M WIRE GAUGES

| gauge (W/M) | dia. (mils) | gauge (W/M) | dia. (mils) |
|---|---|---|---|
| 14 | 80.0 | 26 | 18.1 |
| 14½ | 76.0 | 27 | 17.3 |
| 15 | 72.0 | 28 | 16.2 |
| 15½ | 68.0 | 29 | 15.0 |
| 16 | 62.5 | 30 | 14.0 |
| 16½ | 58.0 | 31 | 13.2 |
| 17 | 54.0 | 32 | 12.8 |
| 17½ | 51.0 | 33 | 11.8 |
| 18 | 47.5 | 34 | 10.4 |
| 18½ | 44.3 | 35 | 9.5 |
| 19 | 41.0 | 36 | 9.0 |
| 19½ | 38.0 | 37 | 8.5 |
| 20 | 34.8 | 38 | 8.0 |
| 21 | 31.8 | 39 | 7.5 |
| 22 | 28.6 | 40 | 7.0 |
| 23 | 25.8 | 41 | 6.6 |
| 24 | 23.0 | 42 | 6.2 |
| 25 | 20.4 | 43 | 6.0 |

Please note that the Washburn and Moen (W/M) gauge numbers bear no resemblance to music wire gauge numbers. Most stringmakers offer copper wrap gauges from #36 through #14 (Mapes also has #13-1/2, which is 84 mils diameter). In addition, Tuners Supply offers #37 and #38 and A. Isaac Pianos offers #37 through #43. Mapes also offers iron with 5% (by weight) copper electroplate, which they call "copper ply on low metaloid steel," in gauges #36 through #15-1/2. Schaff Piano Supply offers aluminum in gauges #24 through #28 and is currently the only supplier of aluminum wound strings of which I am aware.

To the best of my knowledge, stringmakers today make no distinction between iron and copper wraps when duplicating an old set of strings in copper, although, strictly speaking, they should. This is because the weight added to a string by these two wraps is not quite the same for the same wrap gauge. Only the aluminum is significantly different in weight for a given gauge number. The precise equivalence of different wrap materials is a somewhat tricky subject because the weighting due to, say, an iron wrap compared to an aluminum wrap depends not only on the wrap gauge numbers but also on the core used and on the amount of distortion suffered by each wrap as it is being wound onto the core. There are still other factors to consider, including holding power and some more subtle matters, but the principal consideration is simply that they give the same added mass per unit length along the string. In this case, it can be shown that a wrap material of thickness $\mathbf{d_1}$ wound onto a core of diameter $\mathbf{d}$ has an equivalent (alternate) wrap of thickness $\mathbf{d_2}$ given by the formula

$$d_2 = \frac{d}{2}\left[\sqrt{1 + 4\frac{A_1 d_1}{A_2 d}\left(1 + \frac{d_1}{d}\right)} - 1\right]$$

where $A_1$ is the "weighting constant" for the original wrap material and $A_2$ is the "weighting constant" for the equivalent wrap. These two values of the constant **A** are chosen from $A = .89$ (copper), **0.79** (iron, plated or unplated) and **0.27** (aluminum). It is important to keep in mind that the wire dimensions $d_1$ and $d_2$ above are really the wire thicknesses perpendicular to the core after winding. This dimension is always smaller than the original diameter because of the distortion suffered by the wrap as it is wound onto the core. We will assume here that the distortion is about 5%, a number which is fairly typical, although I have seen some values as low as 2% and as high as 30%. By 5% distortion, I mean that the wrap wire (originally of circular cross section) is reduced to about 95% of its original thickness perpendicular to the core wire and increases to approximately 105% of its original thickness along the length of the core wire.

Thus, the cross section of the wrap becomes somewhat elliptical after winding, but the volume ratio of wrap material to adjacent air spaces remains about the same. Because of this fact, the weighting factor

$$B = A\left(\frac{D^2}{d^2} - 1\right)$$

which we have often referred to and calculated, is quite accurate, regardless of the degree of wrap distortion. It is also accurate for double wound strings, because stringmakers almost always choose the underwrap and outer wrap to be sufficiently different in size that there is no nesting of the outer wrap between the turns of the underwrap. Thus, the same volume ratio of wrap material to air spaces is maintained for either single or double wound strings. For your information, the outer wrap is usually two to three times larger in thickness than the inner wrap on double wound strings.

Let us give an example to illustrate the alternate wrap formula. Suppose we have a #19 core (music gauge!) wrapped with #36 copper (W/M gauge!) and we wish to know the equivalent aluminum wrap gauge. Typically, the copper thickness would be 9.0 mils (see Table 8.1), less about 5% due to distortion during winding, which turns out to be 8.6 mils. Thus, we have

$d_1$ = **8.6** mils (after distortion)
$d$ = **43** mils (core)
$A_1$ = **0.89** (copper)
$A_2$ = **0.27** (aluminum)

The formula is therefore calculated as follows:

$$d_2 = \frac{43}{2}\left[\sqrt{1+4\frac{0.89}{0.27}\frac{8.6}{43}\left(1+\frac{8.6}{43}\right)} - 1\right]$$

$$= 21.5\left[\sqrt{1+2.64(1.20)} - 1\right]$$

$$= 21.5[2.04 - 1]$$

$$= 22.4 \text{ mils (after distortion)}$$

If we then add 5% to this value, we will have the diameter of the (undistorted) equivalent aluminum wrap, which is 23.5 mils. This is obviously very close to #24 gauge, as you can see from Table 8.1. Therefore, it should make little difference acoustically whether the 43 mil core is wrapped with #36 copper or #24 aluminum. I personally prefer the lightweight but rugged aluminum wraps to the more fragile copper gauges (#36 through #43) if I am designing the transition from plain to wound unisons on the treble bridge, but you will no doubt want to make your own judgment. There is no reason whatsoever to use aluminum wound strings on the bass bridge because the shorter speaking length of the uppermost bass unisons compared to the lowest treble unison actually requires a heavier wrap for a proper transition.

For those of you who would rather not calculate equivalent wraps, there is a simple rule-of-thumb for finding the copper equivalent of an iron wrap: just add 2 to all iron gauges from #44 through #36 to get the equivalent copper gauges; likewise, add 1 to all iron gauges from #34 through #20 and add 1/2 to all iron gauges from #19 through #14. This rule applies for any core size from #15 through #26 music wire gauge.

The equivalence between aluminum and either copper or iron is a bit trickier. This is given in Table 8.2.

TABLE 8.2. APPROXIMATE ALUMINUM EQUIVALENT OF IRON/COPPER WRAPS

| COPPER (W/M) | core (music gauge) | IRON (W/M) | core (music gauge) | ALUMINUM (W/M) (on same core) |
|---|---|---|---|---|
| 35 | 15 - 17 | 34 | 15 - 19 | |
| 36 | 15 - 23 | 35 | 15 - 23 | 24 |
| 37 | 15 - 23 | 36 | 20½-23 | |
| 38 | 15 - 23 | 36 | 15 - 20 | |
| 39 | 15 23 | 37 | 15 - 23 | 25 |
| | | 38 | 18½-23 | |
| 40 | 15 - 23 | 38 | 15 - 18 | |
| 41 | 17½-23 | 39 | 15 - 23 | 26 |
| 41 | 15 - 17 | | | |
| 42 | 17½-23 | 40 | 15 -23 | 27 |
| 42 | 15 - 17 | 41 | 15 -23 | |
| 43 | 15 - 23 | 42 | 22 -23 | 28 |
| 44 | 15 - 23 | | | |

Remember, we are assuming all wraps suffer roughly a 5% distortion during the winding process, so the actual overall diameter **D** of the wound string will be equal to the core diameter **d** plus twice the (distorted) wrap thickness, i.e.,

$$D = d + 1.9d_w$$

where $d_w$ is the original (undistorted) wrap diameter as given in the first table in this chapter. Suppose, for example, we have an iron wound string of overall diameter **D** = 60 mils and 17-1/2 music wire core (**d** = 40 mils). Turning the above formula around, $d_w = D - d)/1.9$ = 10.5 mils, which is the presumed original diameter of the iron wrap. This is close to W/M gauge #34, so the equivalent aluminum gauge according to Table 8.2 would be #24.

Although you may be tempted to specify the wrap size(s) in W/M gauge when you order strings for a new or modified scale, I have found it safer to specify the core diameter in (decimal) inches and the overall diameter the same way. This way, you talk the stringmaker's language and you also have some recourse if, for some reason, he severely distorts the wrap while winding it onto the core. You will know this has happened if the overall string diameters turn out appreciably smaller than you had calculated based upon the 5% distortion factor.

He might make it over for you anyway, as most stringmakers are nice people and eager to please, but it is better to tell him what you want to end up with and let him decide how he is going to do it. This also applies to double wound strings—let him decide what combination of wraps he will need to arrive at a certain overall diameter, at least until your experience indicates there is a better way.

In calculating these overall diameters, however, I would suggest you use the available wrap gauges indicated in this chapter and assume a 5% distortion will take place, as discussed above.

Our detailed discussion of wound strings started with formulas for tension, loudness, inharmonicity, etc. in both plain and wound unisons. We continued with the design of the unwound segments between the wrap ends and the bridge and agraffe terminations in order to minimize both inharmonicity and wildness in conventionally designed (swaged) wound strings. The length and extent of the swages themselves were factors in this consideration.

We have also discussed the purpose of adding wrap to a string in the first place and the special benefits and problems involved in doing this on the treble bridge in the smaller grands and verticals.

We explained why it is necessary to use very lightweight wraps in making the plain-to-wound string transition on the treble bridge and suggested the possibility of using some of the five commercially available aluminum wraps, W/M gauges #24–#28. These give the same mass weighting as the ultra-fine W/M copper gauges #35–#43, but are more rugged and possess greater holding strength.

One problem with aluminum, however, is possible long-term corrosion effects because of its contact with the steel core, although this is not necessarily a problem peculiar to aluminum. As anyone knows who has seen a plumbing joint between copper and iron fittings, dissimilar metals in contact corrode in the presence of excessive humidity or moisture.

Ordinarily, there would be no appreciable corrosion problem with aluminum or copper on steel, as long as there is reasonable humidity control in the piano. This may not be the case in certain situations and the problem is greatly compounded if there is any salt spray in the air.

In Cleveland, where there is typically a variation in relative humidity from 35% to 60% even in air-conditioned homes, I have not seen any corrosion problems with aluminum strings at least 10 to 15 years old. I mention all of this because one technician I know from Washington (state) seems convinced that aluminum wound strings corrode faster there than copper ones. Perhaps a humidity control system installed at the time of rebuilding/rescaling would help but, unless we get more input from other technicians around the country, the use of aluminum wound strings may remain controversial.

A list was given of the copper, iron and aluminum wrap gauges which are available today from American and Canadian stringmakers. Rules-of-thumb as well as formulas were given to enable a technician to determine equivalent wrap gauges in all three of these wrap materials. It was shown that the determination of such alternate wraps depends not only on the "weighting constant" for each wrap material, but also on the core size used and on the distortion suffered by the different wraps as they are wound onto the core.

Let us now discuss a few more aspects of wound strings, including what happens when a new string is pulled up to pitch, string elongation and a method of predicting how to specify the winding length on a specially designed string so the unwound segments at the agraffe and bridge terminations have predetermined lengths **after** pulling the string up to pitch.

First, what really happens when a wound string is first pulled up to pitch? Does the winding get looser or tighter? Why is the string at first unstable in its tuning? I think we can make some headway on this subject by first noting that a string changes in at least three ways when pulled up to pitch.

First, the string obviously changes its shape at the hitchpin, tuning pin and various bearing points.

Secondly, the string stretches out (elongates).

Thirdly, the string gets smaller in diameter. We can express the *fractional* elongation of the steel core as

$$\mathbf{f = 0.043\ T/d^2}$$

where **d** is the core diameter in mils and **T** is the tension in pounds. The *fractional* decrease in the core diameter is just **f** multiplied by "Poisson's ratio," a number approximately equal to 0.3 for steel.

For instance, consider the two example strings discussed in Chapter 6: (1) a short upper bass string in a small grand having **T** = 170 pounds and **d** = 42 mils; (2) a long string in a concert grand having **T** = 360 pounds and **d** = 63 mils. In the first case, $\mathbf{f} = 0.043 \times 170 \div (42)^2 =$ 0.0041. In the second case, a similar calculation gives 0.0039, about the same. The fractional decrease in diameter of the core would therefore be about $0.004 \times 0.3 =$ 0.0012 in both cases.

If you will recall, the first string had a speaking length **L** = 35", so the elongation in the speaking length would just be **L** x **f** = 0.14", slightly more than 1/8". In the second case, **L** = 80", so **L** x **f** = 0.31", about 5/16".

Even greater is the total elongation of these strings from hitchpin to tuning pin, which is the extra length of wire which winds around the tuning pin when pulling the string up from the slack position. We can express the elongation $E_G$ of any string segment of initial (zero tension) length **G** when pulled up to tension **T** as

$$E_G = (0.043T/d^2)G$$

The symbol **G** could be the entire string length or just the speaking length **L** or any other portion of the whole string. The elongation $E_G$ will have whatever units (inches, centimeters, etc.) you use for **G**, but you must express **T** in pounds and **d** in mils, as before.

The decrease in core diameter for the two example strings above is 0.3 x **f** x **d** = 0.052 mils and 0.074 mils, respectively, which is less than one ten-thousandth of an inch! This is, however, more than nothing at all, so the question arises whether the wrap therefore tends to loosen as the string is pulled up to pitch.

The answer is not obvious because several other subtle factors are also at work. One of these is that the helical wrap turns decrease in diameter, but not enough to make up for the decrease in the core, especially for the lightly wound strings. However, this problem can be remedied by twisting the string in the direction of the wrap turns just prior to installation.

In the case of the two example strings above, I calculate (formulas not given here) that it would take about 1-1/2 turns of the lightweight bass string and about 1/3 turn of the heavy string to make up for the difference in contraction of these respective cores and wraps. These calculated numbers do not take into account that, during the winding, twisting and chipping processes, there are inelastic as well as elastic deformations taking place, but the numbers above are nevertheless roughly in accordance with actual practice.

Although the changes in dimensions discussed above are very small, all it takes is the slightest loss of contact between the wrap and core to make a wound string noisy or lose its liveliness. Another factor causing a dead string is corrosion or debris. When a wound string is pulled up to pitch, the wrap turns separate slightly. This in itself is good, because the spaces between the turns make a wound string more flexible. However, if debris and/or corrosion build up in these spaces, it can not only decrease flexibility (increasing inharmonicity) but also absorb vibratory energy. Sometimes, shaking the debris from such a string will restore some liveliness, but corrosion will likely be left behind because it is concentrated between the core and inside surfaces of the wrap turns for reasons discussed earlier.

When a new string is installed, there is a noticeable tuning instability caused by a slow process of inelastic deformation called "mechanical creep."

This process can only take place in those portions of the string where the elastic limit has been exceeded. Except for the slight curve in the original wire due to coiling it up for packaging and storage, the only regions which have exceeded this limit are near the hitch and tuning pins and various bearing and string rest points.

As a string is first pulled up to pitch, a primary deformation process occurs near these points wherein the wire rapidly undergoes a change in shape due to the tremendous leverage inherent in pulling a wire around a fixed point. Then a slower secondary creep takes place wherein these bends gradually take on sharper definition and the slight curve in the original wire straightens out. This process slows down with time as these bends approach a stable configuration, and the mechanical advantage (leverage) for further bending is thus reduced.

Instability in new strings can be minimized by initially overtensioning via several well known methods. Pulling the strings a prescribed amount above pitch and later lowering to standard pitch is by far the safest method because you can calculate exactly what tensions you have. Most piano strings are between 35% and 50% of their breaking tensions, except at the various bends. Pulling them up 1 semitone increases tension by only 12%, i.e., to 39% to 56% of breaking, which most pianos should easily be able to withstand. Some technicians use pushing, pulling, rolling and rubbing methods on one string at a time, which can easily increase string tension more than just 12% if one is not careful.

Let us finish this discussion of wound strings by giving the string designer's formula for the hitch-to-start-of-winding distance $\mathbf{L_1}$ and the winding length $\mathbf{L_2}$ in a slack string, assuming the speaking length is $\mathbf{L}$, the hitch-to-start-of-winding distance (speaking side!) is $\mathbf{M}$ and you want the unwound segments at the agraffe and bridge ends to be $\mathbf{a}$ and $\mathbf{b}$, respectively, *after* pulling the string up to pitch:

$$\mathbf{L_1 = (M + b)/(1 + f)}$$
$$\mathbf{L_2 = (L - a - b)/(1 + f)}$$

The quantity $\mathbf{f}$ was defined earlier. These formulas take into account that both $\mathbf{L_1}$ and $\mathbf{L_2}$ will get longer as the string is pulled up to pitch. All lengths ($\mathbf{L}$, $\mathbf{L_1}$, $\mathbf{L_2}$, $\mathbf{a}$ and $\mathbf{b}$) should be expressed in the same units (inches or whatever), but $\mathbf{f}$ should be calculated as described previously.

If you are redesigning all the unisons on the bass bridge, an easier way to give the stringmaker the information he needs than measuring all those **M** values and calculating all those $L_1$ and $L_2$ values is to make a paper (rub) pattern showing the hitch and bridge pins and agraffe or capo bar line. Then calculate the $L_1$ and $L_2$ values at the beginning and end of each section (monochords, bichords, etc.) and mark these distances on the paper pattern. Draw a line connecting corresponding points at each end of each section and instruct the stringmaker that these lines represent the ends of the windings *before* the strings are pulled up to pitch.

There are several other interesting aspects of string design and behavior we could discuss here, but I think we should get back to the subject of scale evaluation and modification.

# 9 Scaling Formulas
## (Summary and Utilization)

The reason so much time has been spent on design considerations for wound strings is that this information is groundwork we need before returning to the subject on which we started—piano scale evaluation and modification.

At this point, it would be helpful to summarize the formulas which have been presented so far and also put the scaling rules and comments made up to now in perspective. In Table 9.1, I have listed those physical quantities which we would either measure directly, specify from tables or estimate from experience. Table 9.2 lists those acoustical and mechanical parameters which we would ordinarily have to calculate if we were evaluating or modifying a scale.

One of these quantities, overall string diameter **D**, appears in both Tables 9.1 and 9.2. This would be *measured* if we were just evaluating a scale. However, if we were designing wound strings (modifying a scale), **D** would be *calculated* from the formula in Table 9.2 using available music wire gauges for the core and available W/M wire gauges for the copper, iron and aluminum wraps.

Note that the unwound segments **a** and **b** are also listed in both Tables 9.1 and 9.2. Again, these would just be *measured* if we were evaluating an existing scale.

However, rather than calculate them when designing new wound strings, I suggest simply letting **a** = **b** = 1/2". The calculation formula is only approximate and you should probably not make them less than 1/2" anyhow in order to allow for the stringmaker's tolerance on winding lengths and hitchpin loops and the secondary "creep" process over the life of the string as described in the last chapter.

TABLE 9.1.
MEASURED QUANTITIES IN SCALE EVALUATION/MODIFICATION

| Quantity | Symbol | Units | Quantity | Symbol | Units |
|---|---|---|---|---|---|
| speaking length | **L** | inches | number of strings in the unison | **N** | none |
| any portion of total string length | **G** | inches | agraffe (or capo bar)-to-hammer-strike-point distance | **H** | inches |
| agraffe-to-start-of-winding (string at pitch) | **a** | inches | hitch-to-bridge pin distance (speaking side) | **M** | inches |
| speaking-side-bridge-pin-to start-of-winding (string at pitch) | **b** | inches | number of note as it lies on the scale | **m** | none |
| steel wire dia. | **d** | mils | partial number (1= fundamental) | **n** | none |
| wrap wire dia. (before wrapping) | $d_W$ | mils | | | |
| overall string diameter | **D** | mils | | | |

Note: a mil is one one-thousandth of an inch, i.e. 0.001" inches

For those of you who prefer to work in the metric system, say tension in kilograms and all lengths and diameters in centimeters, we can modify the formulas in Table 9.2 as follows:

- Change the constant **802.6** in the tension (**T**) formula to **7.69**.
- Change the constant **0.557** in the $T_{max}$ formula to **5353**.
- Change the constant **139430** in stiffness formula (**S**) to **0.00000198**.
- Change the constant **0.043** in the $E_G$ and **f** formulas to **0.00000061**.

TABLE 9.2

CALCULATED QUANTITIES IN SCALE EVALUATION/MODIFICATION

| Quantity | Units | Formula |
|---|---|---|
| string tension | lbs | $T = 2^{\left(\frac{m}{6}\right)}\left(\frac{Ld}{802.6}\right)^2\left[1+B\right]$, quantity **B** defined below |
| max. safe string tension | lbs | $T_{max.} = 0.557d^{1.667}$ (60% of breaking strength) |
| wrap weighting factor | none | $B = A\left(\frac{D^2}{d^2}-1\right)$, where A = { 0.89 for copper wrap; 0.79 for iron wrap; 0.27 for aluminum wrap } |
| inharmonicity of $n^{th}$ partial | cents | $I_n = 1731(n^2-1)\left\{S\left(1+\frac{B}{8}\right)+\frac{3B}{1+B}\left[\left\lvert\frac{a}{L}-\sqrt{S}\right\rvert^3+\left\lvert\frac{b}{L}-\sqrt{S}\right\rvert^3\right]\right\}$ |
| steel wire stiffness factor | none | $S = d^4/139430\,L^2T$ (used in $I_n$ formula above and **a** and **b** formulas below) |
| loudness factor | NA | $Z = d\sqrt{NT(1+B)}$ (larger value gives louder, less sustaining tone) |
| hammer/string contact time factor | NA | **NT/H** (larger value gives faster hammer rebound, less damping of higher partials) |
| fractional string elongation | none | $f = 0.043\,T/d^2$ (elastic deformation only) |
| elongation of string segment **G** | inches | $E_G = (0.043\,T/d^2)G$ (elastic deformation only) |
| hitch-to-start-of-winding | inches | $L_1 = (M+b)/(1+f)$ } new, slack strings |
| length of winding | inches | $L_2 = (L-a-b)/(1+f)$ } new, slack strings |
| overall string dia. | mils | $D = d + 1.9\,d_w$ (assumes 5% distortion of wrap) |
| unwound ends **a** and **b** (wrapped string) | inches | $a \approx b \approx L\sqrt{S}$ (approximate only — see text) |

Inharmonicity $I_n$ will still be calculated in cents. Loudness **Z** and the hammer-string contact time factor **NT/H** will have different magnitudes than in the English system, but we are not ordinarily interested in either the units or magnitude of these quantities, just how *smoothly* they change from unison to unison throughout the scale. We might be interested in the magnitudes *per se* only as they compare note for note from one scale to another, but as long as we stick exclusively with either English units or metric units there is no problem in this respect.

Now we come to the question of what to do with all these formulas. I am sure many of you are thinking that it is just not practical to measure all the quantities in Table 9.1 and calculate all the quantities in Table 9.2 for every note on the piano every time we wish to evaluate or modify a scale.

Obviously, none of this makes any sense if it is going to take too much of our valuable time, so let us describe a procedure for cutting the scale evaluation time down to less than one hour plus measurements. Subsequent modification, if necessary, would take a few minutes to an hour longer, depending on the extent of the modification. Would it not be worth that much time to you and your customer to see that the piano you are rebuilding conforms to good scaling practice? Is not improved smoothness in tuning, tone and power an important goal in rebuilding a piano? I think it should be.

The secret to efficient evaluation/modification of a piano scale is first to have a well organized, preprinted worksheet on which you tabulate your measurements (or specified quantities) and also your calculated quantities.

Secondly, you should have a programmable calculator with an efficiently designed program for carrying out these calculations. Part of an example worksheet is given at the end of this chapter, along with a few entries to show you how they might appear.

In this example, the note number on the keyboard (**m** in Table 9.1) appears in the first column, with **A1–E44** on side 1 of the worksheet and **F45–C88** on side 2 (not shown here). The remaining quantities which you would typically key in on your calculator keyboard for each note to be analyzed are listed in the next few columns.

In the 7th column (**N**), the letter **C**, **I**, **A** or **P** following the number of strings in the unison indicates copper, iron, aluminum or no winding (plain), respectively. Placing numbers in columns 5 and 6 when analyzing a scale would be optional. Writing down the wrap gauge numbers in column 6 when modifying a scale is also optional, but handy. The last 5 columns are for the acoustical and mechanical quantities to be calculated (see Table 9.2). Not shown are a few unspecified columns for whatever purpose you wish.

For instance, you could calculate the (slack) lengths $L_1$ and $L_2$ for any wound strings which you may have redesigned (see Table 9.2). Or, you may be interested in string elongation $E_G$ for purposes of evaluating tuning stability (the longer and more uniformly graduated, the better). Or, you might want to compare measured values of **L/H** on either side of the bass/treble break with the generally accepted value of about 8.0 in good scales. You should at least have the same value of this ratio on both sides of this break if different from 8.0; if not, you might want to consider changing your hammer line slightly if this does not introduce other problems you cannot cope with.

Let me state again, as I have in previous chapters, that the principal objective in good scale design is to get $\mathbf{I_4}$, **Z** and **NT/H** to change *smoothly* (in this order of importance) from unison to unison throughout the entire scale, especially across scale breaks such as plain/wound, iron/copper, treble/bass, trichord/bichord, bichord/monochord, etc.

Refer to Chapter 3 for more discussion on this subject, including possible exceptions to the rule.

Also, string tensions should not exceed the maximum safe tension $\mathbf{T_{max}}$ (Table 9.2). Rather than calculate $\mathbf{T_{max}}$ for the worksheet, however, I have chosen to calculate the string tension as a fraction of the breaking tension $\mathbf{T_B}$, where $\mathbf{T_B = T_{max}/0.6}$. Thus $\mathbf{T/T_B}$ in column 12 should not exceed 0.60 and, for most existing scales, will be in the range 0.35 to 0.50 most of the time.

Note the number of significant figures for each column entry on the worksheet. This is important because efficiency demands that you use minimum writing and calculator keystroke motions. Those of you contemplating using a printer with your calculator—do not. I know they are a lot of fun, but stringing out all the measured and/or calculated quantities on a narrow piece of paper tape several feet long is counterproductive. The worksheet is far more efficient for comparing the various acoustical quantities for smoothness from unison to unison.

With experience, one can spot most scaling problems just by looking at the piano, in which case one need only evaluate a small portion of the whole scale. For instance, treble scales are seldom faulty except near the bass end and, at any rate, cannot usually be improved significantly by rescaling the middle and upper registers except for the obvious ploy of inserting any missing half-size wire gauges. On the other hand, the plain/wound transition is very often faulty, even in many otherwise good quality grands, and this problem can usually be spotted visually. Hence, as a practical matter, you need not spend more than one-half hour evaluating a piano scale if you use the general approach I have outlined here.

# WORKSHEET

| PIANO | |
|---|---|
| P | make ____ |
| I | model ____ |
| A | type ____ |
| N | serial ____ |
| O | mfg date ____ |

NOTES

| m | L | sting dia. | | wrap dia. $d_w$ | | N | unwrapped ends (inches) | | $I_4$ | Z | $\frac{T}{T_B}$ | $\frac{NT}{H}$ | T (lbs) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| (key) | inches | d | D | mils | ga. | | a | b | | | | | |
| A1 | 55.7 | 59 | 203 | 75.8 | 14½ | 1C | 1.0 | 0.5 | 8.1 | 2725 | .24 | 29 | 200 |
| A# | 55.0 | 55 | 199 | 75.8 | 14½ | | | | 6.3 | 2737 | .28 | 31 | 211 |
| B3 | 54.3 | 55 | 199 | 71.6 | 15 | | | | 6.1 | 2639 | .29 | 31 | 212 |
| C4 | 53.6 | 51 | 187 | 71.6 | 15 | | | | 4.8 | 2643 | .34 | 33 | 222 |
| C# | 52.9 | 49 | 185 | 71.6 | 15 | | | | 4.1 | 2703 | .39 | 36 | 238 |
| D6 | 52.3 | 47 | 172 | 65.8 | 15½ | | | | 3.6 | 2449 | .40 | 35 | 226 |
| ● | | | | | | | | | | | | | |
| ● | | | | | | | | | | | | | |
| ● | | | | | | | | | | | | | |
| ● | | | | | | | | | | | | | |
| E20 | 42.4 | 38 | 79 | 21.6 | 24 | 2C | 0.8 | 0.5 | 1.9 | 1355 | .40 | 61 | 161 |
| F21 | 47.4 | 37 | 73 | 18.9 | 25 | 2C | 0.8 | 0.5 | 1.2 | 1375 | .51 | 65 | 193 |
| ● | | | | | | | | | | | | | |
| ● | | | | | | | | | | | | | |
| ● | | | | | | | | | | | | | |
| D42 | 22.5 | 39 | 39 | — | — | 3P | — | — | 5.6 | 836 | .37 | 163 | 153 |
| D# | 21.3 | 39 | 39 | — | — | | — | — | 6.2 | 838 | .37 | 173 | 154 |
| E44 | 20.1 | 39 | 39 | — | — | | — | — | 7.0 | 838 | .37 | 184 | 154 |

# 10 Typical Scale Designs and Modifications

In this chapter we will present an example scale evaluation so you can see just what can be expected when you encounter a good scale. There is no way I can choose a piano scale which all of you will agree is a reference standard for the industry, but I probably will not get too many arguments if I choose a Steinway concert grand, in this case a 1923 model D which has both iron and copper wound strings in the bass.

In Table 10.1 I have tabulated calculation results for notes 1–88. Please refer to Chapter 9 if you have trouble remembering the meaning of the symbols or their units. Obviously, this listing of information is not as complete as the example worksheet given there, but it will suffice to illustrate the points which I wish to make. As I indicated earlier, this entire table takes less than one hour to calculate and write down if you use one of the three programmable calculators which are discussed in Appendices 1–3.

Furthermore, recall that one would not ordinarily take the time to calculate and fill in the entire table because most scaling problems are localized and can be spotted visually, as described previously.

For instance, you will find that there are usually no (rectifiable) problems in most of the treble scale if your visual inspection reveals that it conforms reasonably closely to Braid White's rule for treble scaling (Chapter 4). Hence, your attention will most often be drawn to the bass/treble break or the plain/wound break and sometimes the bass scale.

Take a few minutes to examine the table to see just how inharmonicity $\mathbf{I_4}$, loudness **Z**, hammer/string contact-time factor **NT/H**, string tension **T** and speaking length elongation $\mathbf{E_L}$ actually change in a good scale.

Also check the ratio $\mathbf{T/T_B}$, the string tension as a fraction of its breaking strength, to confirm that it always remains below 0.60 (i.e., 60% of breaking tension). Indeed, the strings in this particular scale are conservatively designed at no more than 50% of breaking, although recall from Chapter 8 that the various bent portions of the strings (at the hitch and tuning pins, agraffes, capo bar and string rests) are stressed more severely than this. This is one reason why these strings were designed so conservatively in the first place.

To confirm our previously stated rules for good scale design, you should particularly note the following features of the Steinway scale given here:

- $\mathbf{I_4}$ changes from unison to unison in an almost computer perfect fashion from **A1–C88**, a tribute to the (computerless) men who designed and developed this instrument and evidence for the usefulness of our inharmonicity formula for both plain and wound strings. In particular, note that $\mathbf{I_4}$ decreases at the rate of about 3.0 times per octave, starting at **C88**, leveling out at around 1.0¢ at the bass/treble break (**E20/F21**) and then rising again slightly toward the deep bass.

- **Z** also changes remarkably smoothly from **A1–C88**, decreasing from around 3300 in the bass to about 650 in the high treble. The smoothness in **Z** at the bass/treble break (**E20/F21**) and the copper monochord/iron bichord break (**E8/F9**) attests to the usefulness of our loudness formula **Z** for both plain and wound strings.

- **NT/H** changes smoothly as you progress down the scale, even across the bass/treble break where there are large changes in **T** and **H** individually (see NOTES at the top of the table). But then there are apparently rough transitions at the two remaining breaks in the bass. Actually, one cannot really do much better than this unless the speaking lengths were also to change significantly at these breaks, as they do at the bass/treble break. As mentioned previously, the smoothness in **NT/H** has third priority behind smoothness in $\mathbf{I_4}$ and **Z**. This scale bears out this order in priorities.

- **T** changes in a semi-smooth fashion only within each section of unison types. For instance, in the plain trichord section, string tension decreases in a slightly jagged fashion from around 200 pounds at the low end to about 140 pounds at the treble end. At the bass/treble and monochord/bichord breaks, however, the change in tension is anything but smooth. This illustrates that it is generally incorrect to enforce preconceived notions of "equal tension" on a piano scale. Approaching scale evaluation strictly from such a viewpoint is too simplistic to have any general validity.

- $\mathbf{E_L}$ also changes remarkably smoothly throughout the entire scale, which greatly aids in good (relative) tuning stability.

These observations of a good scale will give you a practical guideline in evaluating other scales.

Some of you may still be concerned whether the Chapter 9 acousto-mechanical formulas can be trusted as a guide to scale evaluation or modification. I developed these formulas over several years and, yes, they have changed slightly since I first started this effort due to additional empirical as well as theoretical inputs and also critiques from a number of pianists and other piano technicians. The evaluation of the Steinway concert grand scale above might seem to be surprisingly conforming, especially in light of the fact that the scale evaluation formulas are indeed relatively simple compared to the enormous acoustical complexity of the piano itself. Although such simple equations cannot possibly account for all the subtle aspects of piano inharmonicity, loudness, etc., nevertheless these formulations do account quite well for those physical phenomena which affect piano sound and tunability in a regular, predictable fashion.

As for the random and virtually unpredictable phenomena, such as bridge movement (particularly that due to soundboard resonances) and non-uniform wires, etc., it is still far more efficient to use the predictable acoustic behavior as a point of departure and then go from there, although I think you will find that it is rarely necessary to introduce any further refinements. Certainly, our evaluation of the Steinway concert grand did not indicate that any significant scale changes are called for, so I do not think you should be too concerned with the

# Table 10.1

| P I A N O | | NOTES |
|---|---|---|
| make | Steinway | |
| model | 'D' (concert) | |
| type | 8'113/4" grand | |
| serial | 219648 | H (20) = 6.75"; L/H = 8.0 |
| mfg date | 1923 | H (21) = 9.0"; L/H = 8.0 |

| m | d | D | N | $l_4$ | Z | $\frac{T}{T_B}$ | $\frac{NT}{H}$ | T | $E_L$ |
|---|---|---|---|---|---|---|---|---|---|
| A1 | 67 | 187 | 1C | 3.0 | 3318 | .34 | 35 | 348 | .27 |
| A# | 67 | 181 | 1C | 2.8 | 3285 | .35 | 37 | 364 | .28 |
| B3 | 63 | 178 | 1C | 2.1 | 3348 | .42 | 40 | 391 | .33 |
| C4 | 59 | 158 | 1C | 1.8 | 2785 | .41 | 35 | 343 | .33 |
| C# | 59 | 152 | 1C | 1.7 | 2721 | .43 | 36 | 353 | .34 |
| D6 | 55 | 140 | 1C | 1.4 | 2428 | .45 | 34 | 332 | .37 |
| D# | 55 | 134 | 1C | 1.3 | 2336 | .45 | 35 | 335 | .37 |
| E8 | 55 | 127 | 1C | 1.3 | 2202 | .45 | 35 | 330 | .36 |
| F9 | 51 | 109 | 2I | 1.3 | 2199 | .37 | 52 | 243 | .30 |
| F# | 51 | 107 | | 1.2 | 2207 | .39 | 55 | 254 | .31 |
| G11 | 48 | 102 | | 1.0 | 2082 | .42 | 55 | 249 | .33 |
| G# | 46 | 96 | | 0.9 | 1919 | .43 | 54 | 238 | .34 |
| A13 | 44 | 87 | 2I | 0.9 | 1641 | .41 | 49 | 211 | .32 |
| A# | 40 | 81 | 3I | 0.7 | 1789 | .44 | 69 | 193 | .35 |
| B15 | 40 | 76 | | 0.8 | 1635 | .42 | 67 | 182 | .32 |
| C16 | 39 | 73 | | 0.8 | 1549 | .42 | 67 | 177 | .31 |
| C# | 39 | 71 | | 0.8 | 1510 | .42 | 70 | 177 | .30 |
| D18 | 38 | 66 | | 0.9 | 1342 | .40 | 66 | 160 | .28 |
| D# | 38 | 63 | | 1.0 | 1254 | .38 | 65 | 153 | .26 |
| E20 | 37 | 62 | 3I | 1.0 | 1228 | .40 | 67 | 151 | .26 |
| F21 | 47 | 47 | 3P | 0.9 | 1155 | .35 | 67 | 201 | .28 |
| F# | 47 | 47 | | 0.9 | 1171 | .36 | 72 | 201 | .27 |
| G23 | 47 | 47 | | 1.0 | 1175 | .37 | 77 | 208 | .26 |
| G# | 47 | 47 | | 1.1 | 1188 | .37 | 82 | 213 | .26 |
| A25 | 45 | 45 | | 1.1 | 1104 | .38 | 81 | 201 | .25 |
| A# | 45 | 45 | | 1.2 | 1113 | .38 | 86 | 204 | .25 |
| B27 | 45 | 45 | | 1.3 | 1121 | .39 | 92 | 207 | .24 |
| C28 | 45 | 45 | | 1.4 | 1121 | .39 | 98 | 207 | .22 |
| C# | 45 | 45 | | 1.6 | 1122 | .39 | | 207 | .21 |
| D30 | 45 | 45 | | 1.8 | 1125 | .39 | | 208 | .20 |
| D# | 43 | 43 | | 1.8 | 1031 | .39 | | 192 | .19 |
| E32 | 43 | 43 | | 2.0 | 1029 | .39 | | 191 | .18 |
| F33 | 43 | 43 | | 2.3 | 1028 | .39 | | 191 | .17 |
| F# | 43 | 43 | | 2.5 | 1032 | .39 | | 192 | .16 |
| G35 | 43 | 43 | | 2.9 | 1028 | .39 | | 190 | .15 |
| G# | 43 | 43 | | 3.2 | 1031 | .39 | | 192 | .14 |
| A37 | 43 | 43 | | 3.6 | 1028 | .39 | | 191 | .14 |
| A# | 41 | 41 | | 3.6 | 938 | .39 | | 175 | .13 |
| B39 | 41 | 41 | | 3.9 | 949 | .39 | | 179 | .13 |
| C40 | 41 | 41 | | 4.4 | 951 | .40 | | 179 | .12 |
| C# | 41 | 41 | | 4.8 | 951 | .40 | | 182 | .11 |
| D42 | 41 | 41 | | 5.3 | 961 | .40 | | 183 | .11 |
| D# | 41 | 41 | | 5.8 | 965 | I41 | | 185 | .11 |
| E44 | 39 | 39 | 3P | 5.8 | 875 | .40 | 168 | 168 | .10 |

| m | d,D | $l_4$ | Z | $\frac{T}{T_B}$ | T |
|---|---|---|---|---|---|
| F45 | 39 | 6.4 | 879 | .41 | 169 |
| F# | 39 | 7.2 | 880 | .41 | 170 |
| G47 | 39 | 8.0 | 882 | .41 | 171 |
| G# | 39 | 9.0 | 882 | .41 | 171 |
| A49 | 39 | 9.7 | 890 | .42 | 174 |
| A# | 39 | 10.9 | 890 | .42 | 174 |
| B51 | 39 | 11.8 | 899 | .43 | 177 |
| C52 | 38 | 12.5 | 854 | .42 | 169 |
| C# | 38 | 13.3 | 865 | .43 | 173 |
| D54 | 38 | 14.8 | 867 | .43 | 174 |
| D# | 38 | 16.1 | 874 | .44 | 176 |
| E56 | 38 | 17.7 | 879 | .45 | 178 |
| F57 | 38 | 19.7 | 881 | .45 | 179 |
| F# | 37 | 21.1 | 834 | .44 | 169 |
| G59 | 37 | 23.1 | 839 | .45 | 171 |
| G# | 37 | 25.6 | 842 | .45 | 172 |
| A61 | 37 | 28.8 | 841 | .45 | 172 |
| A# | 37 | 32.8 | 838 | .45 | 171 |
| B63 | 37 | 37.9 | 832 | .44 | 169 |
| C64 | 36 | 40.0 | 789 | .44 | 160 |
| C# | 36 | 42.4 | 801 | .45 | 165 |
| D66 | 36 | 45.4 | 810 | .46 | 169 |
| D# | 36 | 49.0 | 818 | .47 | 172 |
| E68 | 36 | 53.4 | 824 | .48 | 175 |
| F69 | 36 | 58.9 | 828 | .48 | 176 |
| F# | 36 | 63.2 | 837 | .49 | 180 |
| G71 | 36 | 68.4 | 845 | .50 | 184 |
| G# | 35 | 70.7 | 804 | .50 | 176 |
| A73 | 35 | 81.8 | 798 | .50 | 173 |
| A# | 35 | 91.8 | 798 | .50 | 173 |
| B75 | 35 | 105 | 795 | .49 | 172 |
| C76 | 35 | 115 | 799 | .50 | 174 |
| C# | 34 | 135 | 735 | .47 | 156 |
| D78 | 34 | 143 | 745 | .48 | 160 |
| D# | 34 | 164 | 742 | .48 | 159 |
| E80 | 34 | 178 | 748 | .49 | 161 |
| F81 | 33 | 213 | 683 | .45 | 143 |
| F# | 33 | 258 | 670 | .44 | 137 |
| G83 | 33 | 294 | 667 | .43 | 136 |
| G# | 33 | 312 | 677 | .44 | 140 |
| A85 | 32 | 314 | 644 | .45 | 135 |
| A# | 32 | 338 | 651 | .46 | 138 |
| B87 | 32 | 368 | 656 | .47 | 140 |
| C88 | 32 | 364 | 677 | .50 | 149 |

interest to use the piano's own inharmonicity to do "paper tunings." Jim Coleman, Sr. (Phoenix Chapter) and Dr. Al Sanderson (Boston Chapter) have done some interesting and impressive work in this regard as well as "paper tunings" using an inharmonicity formula.

Now let us examine the scale of a typical small grand and see how it measures up to our rules for good scaling. The first thing I generally do is make a quick examination of the treble scale to see whether it conforms reasonably well to Braid White's rule for wire gauges and speaking lengths (Chapter 4). The original stringing scale is as follows:

| Section #1 | Section #2 | Section #3 |
|---|---|---|
| 4–13-1/2 ga. | 8–16 ga. | 8–18 ga. |
| 4–14 ga. | 4–16-1/2 ga. | 8–18-1/2 ga. |
| 6–14-1/2 ga. | 4–17 ga. | 4–19-1/2 ga. |
| 2–15 ga. | 6–17 1/2 ga. | |
| 4–15-1/2 ga. | | |

As you can see, there is nothing particularly unusual about this scale. If we were to add a couple more unisons of 15 gauge wire and subtract a couple of unisons of 16 gauge wire, then the stringing would conform more closely to White's rule of 5 unisons per half-size music gauge. However, this stringing conforms reasonable well to his rule already, and the speaking lengths are within 3% of those of a concert grand down to about Middle C. Below Middle C, foreshortening of the speaking lengths begins to be more significant due to a reversal of curvature in the treble bridge. This reversal is not as severe as you find in some other small pianos, but it put me on the alert for a potential problem, particularly since there were no wound strings on this treble bridge.

My initial inclination, therefore, was to leave the treble scale as is, except possibly at the bass end where I felt it was prudent to do some calculations backed up by careful listening tests. My philosophy, generally, is to leave a scale alone unless there is really an obvious problem. It is true that many scales were simply "borrowed" from other pianos, possibly with little consideration for the differences in the pianos themselves. Even so, you will find that most treble scales conform fairly closely to Braid White's rule (except at the bass end) and there is a good chance that small deviations from this rule were carried out deliberately in order to compensate for acoustic deficiencies in some other aspects of the piano design (a "dead" spot, etc.).

When I played repeatedly a descending chromatic scale straddling the bass/treble break on the original scale (it had new strings), I noticed a problem common to many pianos smaller than about 7', namely, an increasing stridency as I approached the break, followed by a relative mellowness in the wound unisons. Even if this tonal mismatch could be minimized by a good hammer voicing job, you would still experience trouble tuning or setting an extended temperament across this break. Sound familiar? The problem, as is usually the case, is due to the aforementioned foreshortening of the speaking lengths near the bass end of the treble bridge. This is confirmed by the calculation results summarized in Table 10.2.

Refer to Chapter 9, if you have trouble remembering the meanings of some of the symbols in this table. What I do hope you remember, however, is how smoothly inharmonicity $\mathbf{I_4}$, loudness **Z**, hammer/string contact time factor **NT/H** and speaking length elongation $\mathbf{E_L}$

TABLE 10.2. 5'4" GRAND
NEAR THE BASS/TREBLE BREAK

| m | N | $I_4$ | Z | NT/H | $E_L$ |
|---|---|---|---|---|---|
| 23 | 2C | 2.7¢ | 1485 | 79 | — |
| 24 | 2C | 2.7¢ | 1438 | 81 | — |
| 25 | 3C | 2.8¢ | 1568 | 115 | — |
| 26 | 3C | 2.8¢ | 1546 | 120 | .18 |
| * * | * * | bass/treble break * * * | | | |
| 27 | 3P | 3.8¢ | 809 | 66 | .10 |
| 28 | 3P | 3.7¢ | 838 | 73 | — |
| 29 | 3P | 3.5¢ | 870 | 80 | — |
| 30 | 3P | 3.4¢ | 903 | 88 | — |

change across the bass/treble break in the Steinway concert grand which we analyzed. Not so with this scale. The inharmonicity $\mathbf{I}_4$ jumps 36% across this break compared to no more than 10% for the Steinway grand. Likewise, the loudness factor **Z** and hammer/string contact time factor **NT/H** jump 91% and 82% compared to only 6% and 0%, respectively, for the Steinway. Finally, the speaking length elongation changes 80% across this break, compared to only 10% for the Steinway, which explains the tuning instability problem near the break in the original scale.

I think the root of the problem was lack of understanding of inharmonicity in the late 19th and early 20th centuries, which was the development heyday for the smaller pianos. Even Braid White himself did not have a good handle on this phenomenon until at least the 1940s and, unfortunately, the usual trial and error development techniques, which worked so well for other aspects of piano design (especially in the larger instruments), failed most of the industry in this respect, even some of the biggest names in the business.

It is interesting to note how the manufacturer of this 5'4" grand attempted to minimize the problem we see here. As you can see from the table, he added a third string (i.e., **N = 3**) to the two top bichords on the bass bridge, apparently to increase loudness and decrease hammer/string contact time (larger **NT/H**), both of which would tend to give these lower inharmonicity unisons a little more "oomph" and a little more "edge" to match up tonally with the strident lower treble unisons. This, of course, is not an ideal solution, but it works after a fashion.

Today, we have the understanding of piano acoustics to correct this problem properly. Also, the more progressive manufacturers have been implementing these principles in the design of the newer instruments.

Before we discuss scale modification for this small grand in order to bring it into conformance with modern scaling practices, let me make some comments about Braid White's rule.

Some of you may have judged my offhand acceptance of most of the treble scale in this small grand, just because it conforms reasonably well to Braid White's rule, a cop-out for a "calculating technician." After all, what did Braid White know about inharmonicity and how could such a simple-sounding rule be so universally valid? To answer this point, you should know first of all that Braid White did not make up this rule out of the clear blue. What he did was verbalize an industry consensus in the early 1900s which, of course, represented a tremendous amount of scale development work up to that time and even the virtual perfection of the large (concert-size) instruments. Many variations from this rule were no doubt attempted, but this consensus of many dedicated minds has prevailed even up to the present time with no significant variations that I am aware of, except possibly one which I will now attempt to explain.

As indicated in our evaluation of the Steinway concert grand (very close conformance to Braid White's rule, as you would expect), the inharmonicity in a conventional scale decreases by a factor of about 3.0 for each octave as you proceed down the keyboard until you approach the wound strings. This has some profound implications on a piano's tunability and tone.

For instance, it is impossible in a conventional scale to get the various sets of partials (1-2, 2-4, 3-6; 1-4, 2-8; 1-8, etc.) in the single and multi-octave intervals (or any interval for that matter) to beat (or not beat) the same. This is a consequence of the way in which inharmonicity changes in a conventional scale and has at least one virtue in that it helps to "fuzz-out" the errors inherent in equal temperament.

On the other hand, this situation makes tuning a piano difficult because you cannot simultaneously tune single, double, triple octaves, etc., to be perfectly in tune with themselves or with each other. Instead, the tuner has to compromise using very carefully controlled octave stretching for a first-class job. A good compromise is usually quite acceptable, at least in the larger instruments, but it does require considerable skill.

Now, what would you think if I told you that there is a way to design a piano scale that would allow perfect octave tuning with no such compromising? Impossible? Well, not on paper at least. Theoretically, if you design a scale so that the inharmonicity decreases by a factor of 4.0 per octave instead of a factor of 3.0 per octave as you proceed down the scale, you will have precisely that situation. It is a mathematical fluke in a way, but several present day piano manufacturers have attempted to incorporate this idea into a portion of their treble scales in order to improve tunability. There are several ways to do this, but all require that the treble bridge sweep away from the hammer line more rapidly than in a conventional scale.

For instance, one way would be to change Braid White's rule to state that speaking lengths should increase by 6.08% per unison instead of 5.67% per unison as you proceed down the scale and, at the same time, increase wire gauges by one half size every 12 unisons instead of every 5 unisons. There are other problems with a scheme like this, but the results are interesting if nothing else. The real question is whether such a scheme is really superior to the conventional scale. In principle, it does make the tuner's job easier, but do other sacrifices and problems make this consideration worthwhile? I won't pass judgment here, because I don't have enough experience with this kind of scale, but I thought you would be interested to know that such a scale is possible. Also, I hope by now that you realize that the seemingly simple treble scaling rule verbalized by Braid White has complex implications and should not be trifled with.

Now back to the main problem at hand...how to improve the scale in the bass/treble break region of our small grand. Table 10.3 (1) reviews the situation, with regard to calculated inharmonicity $\mathbf{I_4}$, loudness **Z**, hammer/string contact time factor **NT/H** and speaking length elongation $\mathbf{E_L}$ near this break. If we do a proper job of rescaling then it probably should not be necessary to have that third unison string on the top two notes on the bass bridge. Remember, this was a contrived solution to an inadequate job of scaling in the original design...at least that is my opinion. So let us first remove that 3rd string from A25 and A#26 and see what we are left with. As Table 10.3 (2) shows, the roughness in **Z** and **NT/H** from G#24 to A25 is now eliminated, but there is still considerable roughness across the bass/treble break.

We could try all kinds of solutions, but experience has shown that there is only one simple way to compensate for this kind of foreshortening of proper speaking lengths on the treble bridge, and that is to place some wound unisons on that portion of the treble bridge where the curvature has reversed back toward the keyboard. If there is no prominent point where this foreshortening or reversal begins, I have found that a good place to start is where the calculated inharmonicity $\mathbf{I_4}$ first starts rising as you proceed down the treble scale. Table 10.3 (1) shows that this occurs at D30, where $\mathbf{I_4}$ has risen from 3.1¢ to 3.4¢. Thus our task is to smooth the transition in $\mathbf{I_4}$, **Z**, **NT/H** and $\mathbf{E_L}$ (in this order of importance) as best we can from D#31 to A#26.

Again, experience shows that the best way to deal with this situation is to switch to lightly wound bichords from B27 to D30. For starters, let us leave the steel wire size as is (44 mils) on these four unisons and simply add #36 W/M copper wrap, which is the most common lightweight wrap available. Let us make the unwrapped end segments **a** and **b** equal to 1/2" for the time being. With a programmable calculator, it takes about two to three minutes to calculate and write down $\mathbf{I_4}$, **Z**, **NT/H** and $\mathbf{E_L}$ for these four unisons. The results are shown in Table

10.3 (3).

As you can see, we now have a much improved situation with respect to **Z**, **NT/H**, and $E_L$ across the bass/ treble break, but a jump in $I_4$ remains and we have created some new problems at the new plain/wound transition D30/D#31. We can fix up the A#26/B27 transition by (1) going to a 45 mil core at B27 (increases $I_4$ to 2.4¢ and **Z** to 1247) and (2) increasing the unwound segments **a** and **b** from 0.5" to 1.1" (increases $I_4$ further to 2.8¢, with no effect on anything else). The D30/D#31 transition is more difficult. The loudness **Z** is already too large at D30 and $I_4$ is too small. We can raise $I_4$ a little by increasing **a** and **b**, as before, but if we change the core size one way or the other, we are going to make either $I_4$ or **Z** better at the expense of the other. Clearly we need to do something else.

That something else, it turns out, is to find a lighter wrap for D30. As I previously explained, the best we can do from commercial stringmakers is either a #43 W/M copper wrap or a #28 W/M aluminum wrap, so with a few more minutes of trial and error calculations I finally come to the conclusion that the best D30/D#31 transition occurs with a 43 mil core wrapped with the more rugged #28 W/M aluminum wire, and both **a** and **b** set at 1.2". I do not like to make **a** or **b** over 1", but it is the only choice I have in order to bring $I_4$ at D30 up close to the value of 3.1¢ at D#31.

That's it. Once you have got the plain/wound transition and the bass/treble transition, the rest (C28 and C#29) is easy. The final proposed scale modification actually uses aluminum wraps exclusively (you could mix aluminum and copper if you wished) and this is shown in Table 10.3 (4). Not shown are the new wire tensions (around 200 lbs.) and percent of breaking tension (around 40%), both of which are entirely acceptable.

In fact, the combined tension of each pair of aluminum wrapped bichord strings is about the same as the combined tension in each unison of corresponding original plain trichords.

Hence, we have changed the overall tension in this scale very little with this modification. Most importantly, we have eliminated the 36% jump in $\mathbf{I_4}$ at the bass/treble break and, at the same time, reduced the 91% jump in **Z** and the 82% jump in **NT/H** to only 2% and 4%, respectively. The 80% jump in string elongation at the bass/treble break has been moved and reduced to a 50% jump at the new plain/wound transition D30/D#31. That is about as good as we can hope for unless we are willing to design a tenor bridge for our new wound strings. Then, and only then, can you get all four mechanical/acoustical quantities $\mathbf{I_4}$, **Z**, **NT/H** and $\mathbf{E_L}$ to change in a near perfect fashion across the plain/wound break in a small piano. The Mason & Hamlin AA and A. B. Chase grands are good examples of this.

A close resemblance to the above scale modification was actually carried out on my own piano several years ago when I was using slightly different formulas than I have given you in these articles. Even that modification was judged successful by other piano technicians and pianists, but the refinement in the present formulas is in part a result of those early scaling experiences.

## Table 10.3
## Scale Modification in a 5'4" Grand*

| m | L | d | D | a | b | N | $I_4$ | Z | NT/H | $E_L$ |
|---|---|---|---|---|---|---|---|---|---|---|
| | (1) Original Scale | | | | | | | | | |
| G23 | 38.9 | 41 | 79 | 1.0 | 0.6 | 2C | 2.7¢ | 1485 | 79 | 0.19 |
| G#24 | 38.3 | 41 | 76 | 1.0 | 0.6 | 2C | 2.7¢ | 1438 | 81 | 0.19 |
| A25 | 37.7 | 41 | 70 | 1.0 | 0.6 | 3C | 2.8¢ | 1568 | 115 | 0.18 |
| A#26 | 37.1 | 41 | 68 | 1.0 | 0.6 | 3C | 2.8¢ | 1546 | 120 | 0.18 |
| B27 | 40.7 | 44 | 44 | | | 3P | 3.8¢ | 809 | 66 | 0.10 |
| C28 | 39.8 | 44 | 44 | | | 3P | 3.7¢ | 838 | 73 | 0.11 |
| C#29 | 39.0 | 44 | 44 | | | 3P | 3.5¢ | 870 | 80 | 0.11 |
| D30 | 38.2 | 44 | 44 | | | 3P | 3.4¢ | 903 | 88 | 0.12 |
| D#31 | 37.2 | 42 | 42 | | | 3P | 3.1¢ | 849 | 88 | 0.12 |
| E32 | 36.2 | 42 | 42 | | | 3P | 3.1¢ | 875 | 96 | 0.13 |
| | (2) Remove middle string from A25 and A#26 | | | | | | | | | |
| G#24 | | | | | | 2C | 2.7¢ | 1438 | 81 | 0.19 |
| A25 | | | | | | 2C | 2.8¢ | 1280 | 76 | 0.18 |
| A#26 | | | | | | 2C | 2.8¢ | 1262 | 80 | 0.18 |
| B27 | | | | | | 3P | 3.8¢ | 809 | 66 | 0.10 |
| | (3): Add #36 W/M copper wrap to B27 through D30 | | | | | | | | | |
| A#26 | 37.1 | 41 | 68 | 1.0 | 0.6 | 2C | 2.8¢ | 1262 | 80 | 0.18 |
| B27 | | 44 | 61 | 0.5 | 0.5 | 2C | 2.3¢ | 1206 | 81 | 0.19 |
| C28 | | 44 | 61 | 0.5 | 0.5 | 2C | 2.2¢ | 1250 | 89 | 0.20 |
| C#29 | | 44 | 61 | 0.5 | 0.5 | 2C | 2.1¢ | 1297 | 98 | 0.21 |
| D30 | | 44 | 61 | 0.5 | 0.5 | 2C | 2.1¢ | 1346 | 107 | 0.22 |
| D#31 | | 42 | 42 | | | 3P | 3.1¢ | 849 | 88 | 0.12 |
| | (4) Final Modification — use aluminum wraps | | | | | | | | | |
| G#24 | | 41 | 76 | 1.0 | 0.6 | 2C | 2.7¢ | 1438 | 81 | 0.19 |
| A25 | | 41 | 70 | 1.0 | 0.6 | 2C | 2.8¢ | 1280 | 76 | 0.18 |
| A#26 | | 41 | 68 | 1.0 | 0.6 | 2C | 2.8¢ | 1262 | 80 | 0.18 |
| B27 | | 45 | 89 | 1.1 | 1.1 | 2A | 2.8¢ | 1234 | 83 | 0.18 |
| C28 | | 44 | 83 | 1.1 | 1.1 | 2A | 2.8¢ | 1157 | 82 | 0.18 |
| C#29 | | 44 | 78 | 1.1 | 1.1 | 2A | 2.8¢ | 1121 | 84 | 0.18 |
| D30 | | 43 | 74 | 1.2 | 1.2 | 2A | 2.9¢ | 1077 | 86 | 0.18 |
| D#31 | | 42 | 42 | | | 3P | 3.1¢ | 849 | 88 | 0.12 |

*Refer to Chapter 9 for a complete discussion of the symbols used in this table and also the calculation formulas referred to in the text.

# 11 Author Update

(May, 1990)

I would like to thank the PTG Foundation Press for choosing the "Calculating Technician" articles for its first publication. I am gratified to learn that there is a growing interest in piano scale evaluation and modification and that others have continued to critique and to contribute to this subject.

In these intervening years, perhaps the most significant change for "calculating technicians" has been the easy availability of computers. This is not to say that these machines have rendered calculators obsolete, particularly the programmable ones, and calculators are still much less expensive and more portable. However, computers (and their printers) are faster and have opened the door to more convenient formatting of input and output information. Also, they can be programmed in an easier, higher-level language, such as BASIC. Perhaps even easier to use than BASIC is the popular "spreadsheet" software, which can be programmed to do scientific calculations almost as easily as business forecasting and balance sheets.

I have not personally upgraded any of my calculator programs for computer use, but others in the Guild have. If you inquire around, you can probably find someone who offers such software for a modest fee or possibly at no charge at all.

Along with easy access to high-speed computations has come increased scrutiny of piano scaling formulas, particularly the inharmonicity formula. Some piano technicians have attempted to verify the formula with their own measurements. Results have been mixed. In order to clarify some points of confusion regarding this matter and also to reconcile apparent discrepancies between my formula and one offered by Dr. Albert Sanderson (an often quoted, respected and still active member of the PTG), I offer the following discussion.

A common procedure for determining inharmonicity in a given piano string is to use an electronic tuning aid to measure the "cents" deviation D for each of two octave partials from their respective harmonic values. One then divides the difference of these two readings by the difference of the squares of the partial numbers in order to arrive at an inharmonicity constant **K**, which relates the inharmonicity $I_n$ of any partial **n** to the partial number, as follows (References [7], [8] and [9]):

$$K = (D_n - D_m)/(n^2 - m^2) \text{ cents}$$
$$I_n = K(n^2 - 1) \text{ cents}$$

Note that the inharmonicity of the fundamental ($n = 1$) is implicitly taken to be zero ($1^2 - 1 = 0$) in this definition. There is apparently a great deal of confusion on this issue among "calculating technicians" and even in some authors using $n^2$ instead of $n^2 - 1$ in the $I_n$ formula above. Let me attempt to clarify this point.

If one reads the above cited references very carefully, it is seen that the relation $I_n = Kn^2$ applies only to an ideally stiff string with perfectly "pinned" (hinged) termination. *Real* strings are intermediate between "pinned" and "clamped," which affects the partial frequencies in a *different* and mathematically intractable way. However, this is really of no consequence, because it is possible in a real string to calculate the inharmonicity at partial **n** *relative* to the inharmonicity at the fundamental, whatever that may be. The mathematical complexity due to the terminations drops out in this approach and the $n^2 - 1$ relationship results. This is really all any piano technician would be interested in anyhow, as demonstrated by the fact that one traditionally reports the measurement of partial inharmonicity relative to a zero "cents" setting of the fundamental. In order to convince yourself that the proportionality $n^2 - 1$ (and not $n^2$) is indeed the correct one, measure the inharmonicity of the partials $n = 3,2$ for the A440 string. I think you will see that the ratio of "cents" deviations is much closer to $(3^2 - 1)/(2^2 - 1) = 2.67$ than $3^2/2^2 = 2.25$, assuming this is a fully stabilized string.

One cause for discrepancies when comparing measurements of inharmonicity in wound strings (July 1988 *PTJ*, p. 16) and the formula in Chapter 6 is the presence of an alleged "kink" or "double slope" in the otherwise single-sloped inharmonicity vs. $n^2$ curve for the lower wound strings (Reference [7]). Thus, the practice of using inharmonicity data from higher partials to calculate the slope of inharmonicity vs. $n^2$ for lower partials is valid only for the higher strings on a piano, where this "kink" has progressively disappeared. In other words, only in the plain and very lightly wound strings is this slope (the so called inharmonicity constant **K** or **B**, depending on the author) truly constant.

As an example, Schuck and Young (Reference [7]) have shown that the *indirect* calculation of inharmonicity at the 4th partial in the lower bass, using data from partials $n = 4, 8$, can be as little as one-half the *actual* inharmonicity, as measured directly from partials $n = 1, 4$. My own measurements appear to support this. Without sensitive and highly selective laboratory equipment, however, this behavior is admittedly difficult to verify and the measurements can also be muddled in the "noise" of disturbances due to nonuniform cores and windings, soundboard resonances, imperfect termination conditions, etc. In any case, this perceived behavior is why I modified my inharmonicity formula to give progressively larger inharmonicity values in the bass than predicted by the theory of Miller (Reference [8]), which is often cited by others. I will discuss both Miller's theory and my modification of this theory in a moment, but first I would like to offer some important perspective on this subject of inharmonicity.

Because there are so many complex factors which influence inharmonicity, it should be emphasized that the primary practical value of an inharmonicity formula is not necessarily to calculate highly accurate absolute values of inharmonicity in a given string or strings, but rather to serve as a guideline for evaluating smoothness of inharmonicity changes in stringing scales. This was my goal in *The Calculating Technician*. In this regard, my inharmonicity formula is demonstrated (Chapter 10) to be well suited for scale evaluation across all scale breaks, It can be shown that this would be true with or without the aforementioned formula modification for larger inharmonicity in the bass, so this modification, although certainly a point of interest, should not be a point of contention.

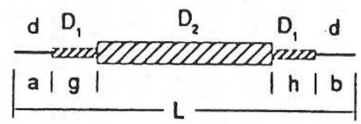
I would also like to note that the emphasis of *The*

*Calculating Technician* formulas is to check smoothness not only in inharmonicity, but also in other important parameters, such as loudness, hammer/string contact time and string elongation. Only if ALL of these acousto-mechanical parameters vary reasonably smoothly across the entire scale are the tuning, volume, voicing, and tuning stability likely to be uniform (or at least smoothly varying) across the many scale breaks (Chapter 10). These include plain-wound transitions, treble-bass bridge transitions, aluminum-iron-copper winding transitions and monochord-bichord-trichord unison transitions. Smoothness in inharmonicity alone is not an adequate guideline. Ultimately, of course, listening tests and good judgment, not calculations alone, should guide any final decisions on rescaling.

With the above perspective in mind, let me discuss in more detail formulas for inharmonicity. Many years ago, I used the often cited article by Miller (Reference [8]) to derive a general expression for inharmonicity in a piano string. I have taken the liberty to simplify and rearrange the formulas in the table on the next page in a form which closely resembles Dr. Albert Sanderson's formulas (which have appeared in the *PTJ* and on PTG convention handout sheets over the years), in order to discuss differences and reconcile apparent discrepancies. In each version, inharmonicity is calculated as shown in the table, where lengths and diameters are in inches, tension is in pounds and sin(x) is in radian mode.

Note first of all that the functional dependence of I-core on the partial number **n** is different in the two versions. As discussed earlier, the standard practical definition of string inharmonicity in the acoustics literature, and the definition implicitly used by any piano

# ROBERTS' VERSION OF MILLER THEORY




ROBERTS VERSION OF MILLER THEORY:

$I_n$ = I-core + I-end-a + I-end-b + I-step-g + I-step-h

I-core = $[(333.8d)^4/TL^2](n^2\text{-}1)$

I-end-a = $137.7[(D_2^2\text{-}d^2)/(D_2^2+.12d^2)][\sin(2\pi a/L) - (1/n)\sin(2\pi na/L)]$

I-end-b = same as I-end-a, except substitute b for a

I-step-g = $137.7[(D_2^2\text{-}D_1^2)/(D_2^2+.12d^2)]x\ [\sin(2\pi(a+g)/L) - \sin(2\pi a/L) - (1/n)(\sin(2\pi n(a+g)/L) - \sin(2\pi na/L))]$

I-step-h = same as I-step-g, except substitute b for a and h for g

SANDERSON VERSION OF MILLER THEORY:

I-core = $[(330d)^4/TL^2]n^2$

I-end-a = $0.287[(D_2^2-d^2)/(D_2^2+.12d^2)][4\sin(4\pi a/L) - \sin(16\pi a/L)]n^2$

I-end-b = same as I-end-a, except substitute b for a

I-step-g = $0.287[(D_2^2-D_1^2)/(D_2^2+.12d^2)]x[4\sin(4\pi(a+g)/L) - 4\sin(4\pi a/L) - \sin(16\pi(a+g)/L) + \sin(16\pi a/L))]n^2$

I-step-h = same as I-step-g, except substitute b for a and h for g technician who has ever measured and reported inharmonicity data, is the one used by Miller and others (References [7], [8], and [9]), where inharmonicity at the nth partial frequency $f_n$ is calculated *relative* to the nth harmonic of the fundamental frequency $f_1$ as

$$I_n = 1200 \times \log_2(f_n/nf_1) \text{cents}$$

This definition automatically leads to the $n^2 - 1$ proportionality for I-core. It should be understood that the appearance of the $n^2$ proportionality in the acoustics literature is an intermediate step in the derivation leading to the inharmonicity definition above. As an aside, I should note that Sanderson's slightly smaller constant, 330 vs. my 333.8, in the formula for I-core results from the use of slightly different values for elastic modulus and density for steel piano wire, which apparently came from different tables. Since we are within about 1% of each other, I would be hard-pressed to say which value is better.

Turning our attention to the inharmonicity contributions from unwound ends and underwrap steps (if any), it should be noted that I-end and I-step do not vary simply as $n^2 - 1$ in Miller's theory, but as a more complex functional dependence on $n$ (see Roberts' version in the table). Sanderson's version lacks this functionality because of an algebraic customization procedure (favoring the relationship between partials $n = 2, 8$) which he has performed on Miller's original formulation (December 1988 *PTJ*, pp. 21-23). Although inharmonicity calculated from this version differs little from the predictions of Miller's formulation in most practical situations of interest, it should be realized by technicians using this formula that the customization procedure has rendered the formula less general than Miller's original version, particularly for partials other than **n** = 8 and long, unwrapped ends.

As an example, consider the 4th partial in a wound **A1** string (note #13) in which **L** = 44.4", **d** = 0.040" and **D** = 0.136" (tension **T** would be 228.6 pounds). If **a** = **b** = 2", then Miller's formula gives $I_4$ = 14¢ vs. Sanderson's $I_4$ = 13¢, reasonably close agreement. However, in the more extreme case **a** = **b** = 3", Miller's $I_4$ = 40¢ vs. Sanderson's $I_4$ = 28¢. Again, I want to reiterate that Sanderson's formula is comparable to Miller's for most practical situations of interest, especially where unwound lengths are less than 2".

When I first presented my inharmonicity formula at the 1978 PTG convention, I felt that the general Miller formula was too formidable (and too much "overkill") to be incorporated into the handout sheets and calculator programs. I therefore used a simplified version corresponding to reasonably short unwrapped ends on the wound strings. However, I also added some complexity by introducing a modification due to Fletcher (Reference [9]) to account for two factors that the Miller formalism does not explicitly take into account: (1) an additional inharmonicity contributed by the flexural inertia of the wrap itself and (2) a reduction in inharmonicity due to the nature of real string terminations—Miller assumed they are "pinned" (i.e., perfectly hinged) whereas Fletcher and others (References [6], [9], and [10]) have shown that real terminations are intermediate between "pinned" and "clamped." These effects are also described in Chapter 6, but their implementation into the "Calculating Technician" inharmonicity formula differs somewhat from that of my 1978 handout sheet.

The 1978 handout used Fletcher's factor "1.07" in the wire stiffness **S** formula and the $\sqrt{S/2}$ term in the inharmonicity formula to account for (1) and (2) above. In the "Calculating Technician" series, I made some changes based upon independent measurements of string inharmonicity performed by Lou Day (Reference [11]) and myself. Our data indicated that (2) above is better described by $\sqrt{S}$ than $\sqrt{S/2}$ and that the factor "1.07" is not nearly enough to account for the larger than expected 4th partial inharmonicity measured in the lower bass strings. I therefore removed the factor of "1.07" from the stiffness formula and placed an (empirically deduced) factor of **(1 + B/8)** next to the core stiffness term **S** in the inharmonicity formula (Chapter 6). This (fudge) factor is the principal difference between my "original" inharmonicity formula and the one given in the "Calculating Technician" series and is the reason why the calculated 4th partial inharmonicity using the Chapter 6 formula rises faster towards the bass than predicted by the Miller formalism.

Although present day "calculating technicians" with computers are probably tempted to use the more complex, general version of Miller's inharmonicity formula (or my doctored version, using inputs from Fletcher and others), I would like to emphasize that the Chapter 6 formula is still entirely adequate for practical situations of interest. A caveat not mentioned in Chapter 6 is that, if you are dealing with a double-wound string, then the measured quantities **a** and **b** should be measured to the start of the *overwrap*, not to the start of the *underwrap*.

For those of you who prefer to use the more general formula due to Miller, but wish to incorporate my modifications due to inputs from Fletcher (Reference [9]), Schuck and Young [7] and Day [11], replace the quantities **a**, **b**, **(a + g)** and **(b + h)** by $|a - L\sqrt{S}|$, $|b - L\sqrt{S}|$, $|a + g - L\sqrt{S}|$ and $|b + h - L\sqrt{S}|$, respectively, in the "Roberts' Version of Miller Theory," in order to account more accurately for actual string termination conditions. In addition, replace the constant 0.12 by $(A^{-1} - 1)$, where $A = 0.89, 0.79, 0.27$ or $0.0$ for copper, iron, aluminum, or no wraps, respectively. Finally, the I-core formula above should be replaced with

$$\text{I-core} = [(333.8d)^4(1 + B/8)/TL^2](n^2 - 1)$$

If you want to "soften" the sometimes controversial fudge factor **(1 + B/8)**, for whatever reason, you can increase the "8" to a larger number or you can replace this factor with another, based on your own inharmonicity measurements in the bass strings. Or, you may wish to eliminate this factor altogether. As indicated previously, the presence or absence of such a factor is really of little consequence with regard to the usefulness of the formula in evaluating smoothness of inharmonicity across a stringing scale.

I hope this discussion satisfactorily clarifies questions which may have arisen over the years due to close scrutiny of various inharmonicity formulas and attempts to verify them experimentally. It would be interesting if someone with laboratory grade equipment could measure inharmonicity vs. $n^2$ in the bass section of different pianos to see whether there is indeed a "kink" in this curve (Reference [7]) and, if so, whether it can be correlated with any particular physical attributes of the scaling or construction. Whatever the outcome, one can always refine (or change altogether) the fudge factor **(1 + B/8)** which is used in this book to modify the calculated inharmonicity of wound strings. If the "kink" is real, it should probably be a function of partial number $n$, and may also be a function of piano size and string termination conditions. All of this is somewhat academic for scale evaluation or modification purposes, but a better grasp of inharmonicity in the wound strings would certainly facilitate "paper" (calculated) tunings, a subject of much interest to several of our calculating technicians.

DR

# Appendix 1

## Calculators for Scaling Computation

Longhand calculation is time consuming. Electronic calculators presently available reduce that time to a minimum. We will attempt to describe the various types of electronic calculators available and how they can be helpful in doing scaling calculations.

There are many electronic calculators for the calculating technician: (1) the simple 4- or 6-function variety; (2) basic scientific calculators; (3) key programmable scientifics; and (4) card programmable scientifics. Some of these may have a built-in or optional printer.

The 4-function variety can perform the four basic arithmetic functions, which are +, –, x, ÷. In addition, they may also be able to do percentages and square roots, which makes them 6-function calculators. The 4- and 6-function units are the most common and usually cost from $5 to $20, depending on features. Printing versions cost more.

One feature to consider is the type of display, usually LED with bright red numbers or LCD, which has dark numbers on a light-colored background. The LED (Light Emitting Diode) display is brighter but, for some people, uncomfortable to look at and causes enough electrical drain on the batteries that you might be replacing batteries several times a year. I would advise not getting a unit with an LED display unless you can get an AC adapter which allows you occasionally to run the calculator from a standard wall outlet while recharging the batteries. In this case, the batteries should last for several years.

An alternative is to get an LCD (Liquid Crystal Display) version. This display has less contrast, but usually is of larger size and uses so little power that the battery should last more than a year.

Another feature to consider is *memory*, which is a place to store (at the touch of a button) some intermediate result you've calculated while you do some additional calculating. The 4- and 6-function calculators usually have just one memory, but they may differ in the ease with which the intermediate result is recalled out of that memory or combined with some other result you've calculated in the meantime. At any rate, these calculators are very easy to use and are adequate for an occasional scientific calculation, as long as the formula requires nothing more than the 4 to 6 functions that the calculator is capable of performing. The tension formula described in Chapter 2 falls into this category, but this will not always be the case.

The basic scientific calculator does everything the 4- or 6-function unit does, plus such functions as $y^x$ (raise any number y to any power x), $\sqrt{}$(square root), $x^2$(square), 1/x (reciprocal) and log or lnx (logarithm of a number). We'll get into why some of these functions are nice to have as we proceed. Basic scientific calculators vary in price from $10 to $50 and the comments made earlier regarding display, AC adapters and memory apply here as well. The more expensive units have more memory and other convenience features and perhaps an AC-adapter at no extra charge.

For the technician who wishes to do calculations on a regular basis, especially if he or she wishes to do identical calculations for each of several unisons in the piano, then the key programmable scientific calculator is the minimum route to go. Such a calculator is used in a manner similar to that of the basic scientific unit; however, its unique feature is that while you are performing some calculation for the first time, it "remembers" what sequence of buttons were pushed on the keyboard to arrive at the answer. Then, for the next unison, you have only to "key in" numerical values for speaking length, wire diameter or whatever is required, and the calculator will automatically carry out all the calculation steps which you did manually the first time through. Also it will do this faster than you could do it yourself, usually in a second or two.

These units vary in price from $35 to $150, reflecting different amounts of "program memory" (i.e., how many different formulas can the calculator remember at one time) and "storage memory" (i.e., how many places are there to store input numbers such as wire lengths, etc. and also intermediate results during the course of the calculations). The price spread may also reflect overall quality and attention to detail, such as the "feel" of the keyboard buttons. Some even "remember" the formulas after you've turned them off (continuous memory), so that you don't even have to go through the initial (first time through) manual calculation the next time you use the calculator, assuming you want to continue with the same type of calculation. If not, you can instruct the calculator to "forget" those formulas.

Finally, there is the card programmable scientific calculator ($230 and up). These are also key programmable but, in addition, the formulas (or programs) can be stored on a small magnetic card which the calculator is capable of "reading." If you want to do some particular type of calculation, say tensions, you just insert the appropriate magnetic program card in a special slot, key in numerical values of wire length, diameter and pitch for a particular unison, and press a "run" button. In a second or two the answer (tension) will be displayed. What could be simpler? You don't even have to understand a thing about arithmetic or formulas if someone else makes up the program for you.

Having discussed different types of electronic calculators available to the piano technician, including the simple 4- or 6-function variety, the basic scientific units and the card and/or key programmable scientifics, we will now go into some detail about specific brands and models. I will restrict this discussion to scientific and programmable calculators made by the two leaders in the field, Texas Instruments (TI) and Hewlett-Packard (HP).

It is difficult to keep abreast in this fast-paced consumer market, because the number of models and prices keep changing so fast. One thing that has not changed is that there are many loyal advocates of both TI and HP calculators, even though differences in the internal logic systems of the two brands require somewhat different approaches to problem solving. Without going into detail, suffice it to say that either approach will work just fine for the calculating piano technician. If I had to make an educated guess (I am familiar with both systems) I'd say that, for the average technician, TI's so-called "Algebraic Operating System" (AOS) initially seems the easiest to use for solving relatively simple formulas. On the other hand, for formulas like our tension equation, and especially even more complicated formulas, I find problem solving somewhat easier using HP's so-called "Reverse Polish Notation" (RPN) logic system. Further, HP's efficient RPN system is complemented by a simpler keyboard layout (fewer keys and simpler labeling of key functions) and a more straightforward system of utilizing combination (merged) keystrokes to define various kinds of math and program functions. I emphasize, however, that either AOS or RPN can be learned by the average person in a short time. Neither system requires any prior experience with electronics, computers or complex mathematics.

If you already have a 4- or 6-function calculator, its logic system most likely resembles AOS rather than RPN, but this should not necessarily deter you from considering HP if you want to move up to a more advanced calculator. I personally prefer HP products because of the RPN logic, exceptional quality and attention to detail ("feel" of the keyboard buttons, etc.), but TI prices are difficult to beat. It is also of interest to note that Dr. Albert Sanderson has successfully implemented the TI-59 card programmable calculator into the evaluation portion of the standardized Guild tuning test. This calculator is well suited to the task because it allows you the flexibility of apportioning the total calculator memory into your choice of "program memory" and "storage memory." It is also far less expensive than the only HP calculator which offers this same feature (HP-41C).

Listed in the following table are most of the current models (as of December 1979) of TI and HP scientific calculators together with some features and discount store prices.

| | make/model | display | "storage" memories | "program" memories | price | printer |
|---|---|---|---|---|---|---|
| Basic Scientific | TI-30 | LED | 1 | none | $ 14 | NA |
| | TI-25 | LCD | 1 | none | $ 30 | NA |
| | TI-50 | LCD | 2 | none | $ 30 | NA |
| | HP-31E | LED | 4 | none | $ 40 | NA |
| Key Programmable | TI-55 | LED | 10 | 32 | $ 35 | NA |
| | TI-57 | LED | 8 | 50 | $ 40 | NA |
| | HP-33E | LED | 8 | 49 | $ 80 | NA |
| | TI-58C | LED | 30 | 240 | $100 | $170 |
| | HP-29C | LED | 30 | 98 | $150 | NA |
| Key & Card Programmable | TI-59 | LED | 60 | 480 | $230 | $170 |
| | HP-67 | LED | 26 | 224 | $350 | NA |
| | HP-97 | LED | 26 | 224 | $630 | INCL |
| | HP-41C | LCD | 17 | 322 | $470 | |
| | plus 1 add-on module | | 49 | 546 | $510 | $330 |
| | plus 4 add-on modules | | 145 | 1218 | $630 | |

Except for the TI-30, all have either a liquid crystal display (LCD) or else a light-emitting diode (LED) display with AC adapter/charger included. The programmable models with a "C" suffix have continuous memory which makes the key programmable versions almost as handy to use as a card programmable, as long as you intend to use the same program over and over. With the HP-41C, it means you may choose to forego the (separate) card reader and save $180 on the prices given in the table. With the TI-58C and TI-59, you can "trade-off" each "storage memory" for eight "program memories" (seven with the HP-41C), or vice versa, which gives these programmable calculators exceptional flexibility for custom requirements.

This is not meant to be an in-depth comparison of the different brands and models available. Other features should be considered before deciding what is best for you. You should be aware, however, that it is sometimes difficult to compare feature for feature in the different brands. One somewhat subtle example is the comparison of "program memory." Since several math and program functions in the TI calculators take two to three times as many "program memories" as an HP calculator would, one should perhaps halve the number of TI "program memories" shown in the table before making the comparison with HP calculators. Another difficult comparison is HP's accessible "stack" and "last x" memories with TI's inaccessible "stack" memories and parentheses notation. These constitute the heart of the RPN and AOS logic systems, and I repeat that both can easily be learned by the average person.

# Appendix 2
## Advantages of Programmable Calculators

To demonstrate the advantages of programmable calculators, let us use the calculation of string tensions as our example.

If we have a copper wound piano string whose speaking length **L** is expressed in inches, core diameter **d** and overall diameter **D** expressed in mils, and pitch **P** expressed in Hertz, then the tension **T** can be calculated:

$$T = \left(\frac{PLd}{20833}\right)^2 \left[1 + 0.89\left(\frac{D^2}{d^2} - 1\right)\right]$$

If your programmable calculator is programmed to calculate tensions, you would typically key in (as on a typewriter keyboard) the numerical values of **P**, **L**, **d** and **D**. However, instead of pushing a "comma" or "space" key between numbers as you would on a typewriter, you would press an "enter" button or a "run/stop" button or perhaps a "store into memory" button on the calculator keyboard, depending on the make and model you have purchased. The time it takes you to do this is roughly the time it would take you to type the same set of numbers on a typewriter keyboard. Having done this, you merely push a "run" button and the calculator does all the adding, subtracting, multiplying and dividing automatically, and displays the answer (tension) after about one second. Then you continue in a like fashion with the next unison, and so on.

Another time-saving feature of the programmable calculator is the $y^x$ button. With this button, you can raise any number **y** to any power **x**. For instance, the number 2 to the power 2 is just 2 squared, which is 4. This is easy, of course, but imagine trying to raise 2 to a power such as 8.167. There is simply no practical way to do this except to use the $y^x$ button on an electronic calculator. I point this out because, by using the $y^x$ button, you can avoid the somewhat time-consuming process of looking up the pitch **P** in a table prior to calculating string tension.

How? It turns out that there is another formula for string tension which is similar to the one we have been using. However, instead of requiring a knowledge of the pitch **P**, the alternate formula requires only a knowledge of the unison number **m**; i.e., the number of the note as it lies on the keyboard. Of course, you still have to know the values of **L**, **d** and **D**. This new formula is written as follows:

$$\mathbf{T} = 2^{\left(\frac{\mathbf{m}}{6}\right)}\left(\frac{\mathbf{Ld}}{802.6}\right)^2\left[1 + 0.89\left(\frac{\mathbf{D}^2}{\mathbf{d}^2} - 1\right)\right]$$

Notice there is no pitch **P** in this formula, but there is the number **m** in the power (or exponent) of 2. This formula adequately predicts the tension for the strings in each of the 88 unisons (i.e., **m** = 1 through 88), provided the piano is tuned to standard pitch and has the standard 88-note keyboard.

Some of you may think this formula looks more complicated than our original tension formula, but a programmable calculator does not think so. The savings in time to you is that you now have only to key in **m**, **L**, **d** and **D** instead of **P**, **L**, **d** and **D**. For instance, for C88 (i.e., **m** = 88) it takes less time to key in the two-digit number

88 than to look up the pitch for C88 in a table (approximately 4186 Hz), and then key in this four-digit number on the calculator keyboard.

Let us illustrate the use of this alternate tension formula. Using the Bechstein string discussed in Chapter 2, the unison number for **F1** is **m** = 9, the speaking length **L** is 75 inches, the core diameter **d** is 63 mils and the overall diameter **D** is 145 mils. In order to calculate tension using the alternate formula, you first calculate

$2^{\left(\frac{m}{6}\right)}$ then $\left(\frac{Ld}{802.6}\right)^2$ and finally $\left[1+0.89\left(\frac{D^2}{d^2}-1\right)\right]$.

Next you multiply these three results together, as explained previously.

We have already explained how to calculate the quantity in square brackets in Chapter 2. The answer is 4.83.

The squared quantity in parentheses $\left(\frac{Ld}{802.6}\right)^2$

is similar to the quantity $\left(\frac{PLd}{208.33}\right)^2$

which appeared in our original tension formula, and is

$$\left(\frac{Ld}{802.6}\right)^2=\left(\frac{75\times 63}{802.6}\right)^2=\left(\frac{4725}{802.6}\right)^2=(589)^2=34.7$$

Finally, the quantity $2^{\left(\frac{m}{6}\right)}$ is calculated by first calculating the exponent (or power) m/6, which is 9 ÷ 6 = 1.5; then use the $y^x$ button on your calculator to find 2 to the power 1.5. On a Texas Instruments calculator, you would key in the 2, then push the $y^x$ button, then key in the 1.5 and finally push the "**equals**" button. The answer 2.828 would then appear in the display. The string tension is therefore **T** = (2.828......) x (34.7) x (4.83) = 474 pounds which is the same answer we got using the original tension formula in Chapter 2.

While it may not seem as if much time is saved in this example calculation using the alternate formula, try doing two or three dozen calculations or more. You will appreciate the $y^x$ button on your programmable calculator. The alternate formula does not save you much time if the calculator is not programmable, but it does save you the aggravation of looking up the pitch for every unison you want to analyze.

I would suggest using an English micrometer or good quality dial caliper for measuring diameters **d** and **D**. Speaking length **L** should be measured with a steel tape subdivided into tenths of an inch rather than sixteenths or thirtyseconds, so you do not have to convert fractions to decimals before keying in the string lengths on the calculator keyboard. Always be efficient and organized in your measurements and recording of data; otherwise, you lose the time advantage which you gained by investing in your electronic calculator.

# Appendix 3

## Programming Scaling Formulas

I have described a general approach for efficient piano scale evaluation/modification and offer program listings for the TI-59 and HP-67 programmable calculators in order to carry out this approach.

Let us also include the HP-41C (recently $260). It is currently the most powerful hand-held calculator available and is the only one with a liquid crystal display, so the batteries will last a very long time and there is no cord to plug in. Also, it has "continuous-memory," which means you do not need the extra cost magnetic card reader if you use it exclusively for piano scale work.

The TI-59 and HP-67 have built-in card readers and recently were priced at $220 and $300, respectively.

For those of you who are still intimidated by the thought of doing all the calculations summarized in Chapter 9, let me assure you that all you have to be able to do is find a button on the calculator keyboard if I tell you the row/column location. With a programmable calculator, no math background is required as long as someone gives you the program listing.

In a nutshell, this is all you typically would have to do to evaluate a scale:

- Convince Santa to get you a TI-59, HP-67 or HP-41C.

- Plug it in (if TI-59 or HP-67) and turn it on.
- Push a sequence of buttons which I will give you (one time only!).
- Key in certain numerical information, as on a typewriter keyboard (string length, diameter, etc.) for a unison of interest.
- Press an appropriate button in the top row of buttons on the calculator keyboard, which I will describe to you.
- Wait about 10 seconds while the calculator calculates all those formulas given in Chapter 9.
- Write down on your worksheet the calculated values of inharmonicity, loudness, tension, etc., in the order that they automatically appear in the calculator display.
- Go on to the next unison.

All this should take less than 1 hour (plus the one time only keying in of the program, step 3 above), even if you evaluate the entire stringing scale. If the inharmonicity ($\mathbf{I_4}$), loudness (**Z**) and hammer/string contact time factor (**NT/H**) all appear to change smoothly from unison to unison on your worksheet, even across all scale breaks, then you can feel reasonably comfortable about rebuilding your piano using its original stringing scale.

If there are some rough spots, all you have to do is repeat the calculations for those particular unisons using different values of wire diameters **d** and/or $\mathbf{d_w}$ and possibly different **a**, **b**, **N** or **A** values (see definitions in Chapter 9). With a little practice, you will quickly zero in on optimum values for smoothing rough spots in the scale. Either that or you will find you really cannot make these spots any better than they already are. For those of you who would dare to do more than just smooth the scale (i.e., lower average inharmonicity in the bass or increase average loudness), I do not advise it unless you have practiced this sort of thing on willing subjects or your own pianos. Yes, I do some of this myself, but let me caution you to be very careful in this regard.

For instance, if your modification causes average tension to change significantly in more than just a few unisons, are you sure you know what impact this will have on downbearing and soundboard motion? You must keep in mind that the formulas which I have given you, as complex as they may appear, are in reality rather simplistic compared to the enormous acoustical complexity of the piano itself.

Even so, I believe the approach I have outlined thus far is a major improvement in scale evaluation/modification over anything previously published or presented to our membership.

Let me now elaborate on the process of using the calculator. I will start with the HP-67 because I find it the easiest to explain. After you have plugged it in, flip the OFF/ON switch to ON and the PRGM/RUN switch to PRGM. The LED display will now show 000.

To key the program into the calculator, you simply follow the sequence of 2-digit keycodes (row/column) listed in Table I. Each step may contain 1, 2 or 3 keystrokes. For instance, to key in the first program step, you need to press three keys; row 3/column 1; row 2/column 5; row 1/column 1. After you press that third key, the display will suddenly change to 001 31 25 11, indicating that you have completed the first program step (001) correctly.

Proceed in like fashion until all 224 steps are completed. You will be happy to know that you will never have to do this again, because the program you have just keyed into the calculator can be stored permanently on a small magnetic card included with your calculator. Should you want to do scale work again a week from now, the calculator can "read" that little card in a few seconds.

Incidentally, there is one exception to the simple keycode designation described above. If a code starts with zero (i.e., 01, 02, etc.) then just press the key having the 2nd code digit printed on it. For instance, program step 002 has the keycodes 33 06. Therefore, you would press the key in row 3/column 3 followed by the key with the 6 printed on it, whereupon the display will change to 002 33 06.

After completing step 224, flip the PRGM/RUN switch to RUN and you are ready to evaluate/modify a scale. You have 9 calculation "routines" at your disposal for this purpose, labelled *A, B, C, D, E, a, b, c* and *d*.

To do a particular routine, just key in the data required, as summarized in Table II. This just indicates that you press the 1st key in the 4th row (called the ENTER key) between numerical entries, just as you would use the "space" or "comma" key on a typewriter to separate a series of numbers. If only one number is to be keyed in, as in routine *D*, then there is no need to push the ENTER key.

Regardless of whether you are going to do an evaluation or a modification, you must first do the *A* routine so the calculator will know the values (or proposed values) of **a, b, N** and **A** and the section of the scale

TABLE A3.1. HP-67 PROGRAM LISTING

| Step | Keycode | Step | Keycode | Step | Keycode | Step | Keycode | Step | Keycode | Step | Keycode |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 001 | 31 25 11 | 039 | 01 | 076 | 31 25 03 | 114 | 35 34 | 151 | 35 34 | 189 | 23 00 |
| 002 | 33 06 | 040 | 06 | 077 | 23 01 | 115 | 71 | 152 | 01 | 190 | 35 72 |
| 003 | 35 53 | 041 | 71 | 078 | 35 72 | 116 | 34 14 | 153 | 51 | 191 | 34 04 |
| 004 | 33 07 | 042 | 31 24 | 079 | 31 25 05 | 117 | 32 54 | 154 | 08 | 192 | 35 22 |
| 005 | 35 53 | 043 | 01 | 080 | 34 00 | 118 | 33 81 09 | 155 | 33 71 03 | 193 | 31 25 12 |
| 006 | 31 25 01 | 044 | 06 | 081 | 34 14 | 119 | 71 | 156 | 81 | 194 | 31 22 02 |
| 007 | 33 12 | 045 | 81 | 082 | 81 | 120 | 31 54 | 157 | 01 | 195 | 35 52 |
| 008 | 35 53 | 046 | 31 83 | 083 | 32 54 | 121 | 33 01 | 158 | 61 | 196 | 51 |
| 009 | 33 11 | 047 | 35 72 | 084 | 01 | 122 | 34 14 | 159 | 71 | 197 | 01 |
| 010 | 44 | 048 | 35 82 | 085 | 51 | 123 | 05 | 160 | 34 11 | 198 | 83 |
| 011 | 35 22 | 049 | 32 83 | 086 | 34 06 | 124 | 41 | 161 | 31 22 04 | 199 | 09 |
| 012 | 31 25 02 | 050 | 01 | 087 | 71 | 125 | 03 | 162 | 34 12 | 200 | 81 |
| 013 | 33 13 | 051 | 06 | 088 | 01 | 126 | 81 | 163 | 31 22 04 | 201 | 33 13 |
| 014 | 33 00 | 052 | 71 | 089 | 61 | 127 | 35 63 | 164 | 61 | 202 | 22 03 |
| 015 | 35 53 | 053 | 31 84 | 090 | 35 33 | 128 | 83 | 165 | 35 34 | 203 | 31 25 14 |
| 016 | 33 14 | 054 | 35 22 | 091 | 34 15 | 129 | 09 | 166 | 81 | 204 | 35 61 03 |
| 017 | 35 53 | 055 | 32 25 11 | 092 | 34 14 | 130 | 03 | 167 | 35 34 | 205 | 34 12 |
| 018 | 33 15 | 056 | 34 13 | 093 | 71 | 131 | 71 | 168 | 01 | 206 | 61 |
| 019 | 35 53 | 057 | 32 25 13 | 094 | 08 | 132 | 33 81 02 | 169 | 51 | 207 | 31 22 06 |
| 020 | 33 08 | 058 | 34 08 | 095 | 00 | 133 | 02 | 170 | 71 | 208 | 34 15 |
| 021 | 35 53 | 059 | 34 15 | 096 | 02 | 134 | 03 | 171 | 03 | 209 | 34 12 |
| 022 | 35 22 | 060 | 35 53 | 097 | 83 | 135 | 33 81 09 | 172 | 71 | 210 | 51 |
| 023 | 31 25 04 | 061 | 35 53 | 098 | 06 | 136 | 34 14 | 173 | 61 | 211 | 34 11 |
| 024 | 34 15 | 062 | 22 13 | 099 | 81 | 137 | 32 54 | 174 | 02 | 212 | 51 |
| 025 | 81 | 063 | 32 25 12 | 100 | 32 54 | 138 | 32 54 | 175 | 06 | 213 | 22 06 |
| 026 | 34 05 | 064 | 34 08 | 101 | 71 | 139 | 34 15 | 176 | 43 | 214 | 32 25 14 |
| 027 | 31 54 | 065 | 34 15 | 102 | 02 | 140 | 33 81 03 | 177 | 03 | 215 | 31 22 01 |
| 028 | 51 | 066 | 34 14 | 103 | 34 08 | 141 | 32 54 | 178 | 71 | 216 | 22 05 |
| 029 | 35 64 | 067 | 35 54 | 104 | 06 | 142 | 81 | 179 | 35 61 03 | 217 | 31 25 15 |
| 030 | 03 | 068 | 31 25 13 | 105 | 81 | 143 | 34 04 | 180 | 23 01 | 218 | 34 15 |
| 031 | 35 63 | 069 | 31 22 02 | 106 | 35 63 | 144 | 81 | 181 | 35 72 | 219 | 35 71 03 |
| 032 | 35 22 | 070 | 01 | 107 | 71 | 145 | 03 | 182 | 34 01 | 220 | 35 52 |
| 033 | 31 25 06 | 071 | 83 | 108 | 33 02 | 146 | 07 | 183 | 23 00 | 221 | 34 09 |
| 034 | 34 09 | 072 | 09 | 109 | 33 04 | 147 | 03 | 184 | 31 84 | 222 | 71 |
| 035 | 01 | 073 | 71 | 110 | 33 09 | 148 | 32 54 | 185 | 34 02 | 223 | 23 02 |
| 036 | 61 | 074 | 61 | 111 | 34 07 | 149 | 81 | 186 | 23 02 | 224 | 35 22 |
| 037 | 81 | 075 | 33 00 | 112 | 71 | 150 | 33 05 | 187 | 84 | | |
| 038 | 23 00 | | | 113 | 33 03 | | | 188 | 34 03 | | |

in which you are interested. Even if you are working in the plain wire section, where **a**, **b** and **A** have no physical significance, you still have to go through the motions of keying in **a** $\uparrow$ **b** $\uparrow$ **N** $\uparrow$ **A**, in that order, so why not just make it **0** $\uparrow$ **0** $\uparrow$ **N** $\uparrow$ **0**? The only times you will have to use this routine again is just before you start working on a section of the scale where one or more of the quantities **a**, **b**, **N** or **A** is changed (not very often).

Incidentally, when evaluating a scale, do not bother changing **a** and **b** for each unison if they only vary by, say, plus or minus 1/8" from unison to unison.

Routine *B* is the principal scale evaluation/ modification routine. If you are in the plain wire section, just input **m** ↑ **L** ↑ **d** ↑ **d** or **m** ↑ **L** ↑ **d** ↑ (either way works) before pressing the *B* key. After a few seconds, the calculator will display $\mathbf{d_w}$, $\mathbf{I_4}$, **Z** and $\mathbf{T/T_B}$, in that order, giving you just enough time to write these values down on your worksheet (see Chapter 9). If you are also interested in **NT/H** and **T**, just press the key in the lower right-hand corner (row 8/column 4) after $\mathbf{T/T_B}$ appears in the display.

Routine *C* is the wound-string scale modification routine. If your initial guess at modified values for **d** and $\mathbf{d_w}$ do not give you the smoothing you have hoped for, you can use routines *a, b, c* and *d* to make further guesses. If all else fails, you may want to propose a major change, say in **N** and/or **A**, in which case you will first have to run the *A* routine again.

The *D* routine would only be run after you are sure of your new scale and *must* be preceded by routine *B, C, a, b, c* or *d*. The display will flash the integer part of $\mathbf{L_1}$ for 1 second, then the number of 16ths for 5 seconds, then repeat this for $\mathbf{L_2}$. This is done so you can express the fractional part of $\mathbf{L_1}$ and $\mathbf{L_2}$ in 16ths of an inch, which most (U.S.) stringmakers prefer.

Routine *E* may not be used much, but it is there if you want it. For optimum tuning stability, elongation should change smoothly from unison to unison. The only way this can happen at the plain/wound break, and still maintain smooth $\mathbf{I_4}$, **Z** and **NT/H**, is to have the 1st wound unison at least 20% shorter than the adjacent plain unison. This is only possible on the larger pianos, unless a separate (tenor) bridge is used for wound treble strings.

TABLE A3.2. HP-67 CALCULATION ROUTINES

| Routine | Input | Description |
|---|---|---|
| *A* | A ↑ b ↑ N ↑ A | Stores a, b, N and A into internal memory for later use. Displays zero when finished. |
| *B* | m ↑ L ↑ d ↑ D | Calculates and displays $d_w$, $l_4$, Z and $T/T_B$; also NT/H and T if desired. |
| C | m ↑ L ↑ d ↑ $d_w$ | Calculates and displays D, $l_4$, Z and $T/T_B$; also NT/H and T if desired. |
| *D* | M | Calculates and displays $L_1$ and $L_2$. |
| *E* | no entry<br>G | Calc. & display $E_L$.<br>Calc. & display $E_G$. |
| *a* | d | Repeats routine *C* changing only d |
| b | $d_w$ | Repeats routine *C* changing only $d_w$. |
| c | d ↑ $d_w$ | Repeats routine *C* changing only d & $d_w$ |
| *d* | a ↑ b | Repeats routine *C* changing only a & b |
| e | — | no routine for this label. |

Routine piano scale evaluation or modification requires: (1) a well-organized preprinted worksheet on which you tabulate your measured and calculated quantities; and (2) a programmable calculator with an efficiently designed program for carrying out these calculations.

An example worksheet has been given in Chapter 9. Since calculators "prefer" that you key in numerical data with fractions expressed in the decimal system, as shown in the example worksheet, I suggest that you purchase a steel tape rule graduated in tenths of an inch rather than sixteenths. Lufkin makes one (#9212-X) and it should be available through your local hardware or industrial tool supply store. This will save time and mistakes in otherwise attempting to convert from English fractional inches to decimal inches. Of course, a metric rule or micrometer and even an English micrometer already read in the decimal fraction system.

I have just described the programming of Chapter 9 formulas into the Hewlett-Packard model HP-67, one of three programmable calculators which I recommend that you consider for this task. The actual programming procedure is easiest to explain for the HP-67, but the equally easy to use HP-41C and the Texas Instruments model TI-59 have more program memory and thus have the possibility for extra program conveniences. Although I do not personally find such extras necessary, some people may want or need them, so I have developed a more extensive program for the HP-41C.

Lou Day of the Denver Chapter of the Guild has worked out a similar program for the TI-59. These programs do the same calculations as the HP-67 program, but they also incorporate a number of helpful "prompts." Prompts are messages or symbols which appear in the calculator display before you key in required measurement data, indicating what data needs to be keyed in. These messages may also appear before or during the display of the calculated quantities, indicating what is being displayed at any given time. This is very much like the automated bank teller systems which ask you questions and give instructions on a TV screen, thus leading you through your transactions. The HP-41C has a $260 base price, but requires an added $40 plug-in memory module (up to 4 are possible) for the program to be discussed here. This module gives it more program memory than the $220 TI-59, so the HP-41C can do more extensive prompting. The HP-41C, unlike either the HP-67 or the TI-59, can display letters, words and a variety of symbols as well as numbers in its display, so the prompts can be more descriptive. Also, the units (inches, mils, pounds, etc.) for the requested data or calculated numbers can be flashed in the display along with numerical values. This is incredible versatility for a handheld calculator, but I am sure we are seeing only the beginning of many such electronic marvels to come.

The programming of the TI-59 and HP-41C is similar to that of the HP-67 but with certain complications. For one thing, these programs are longer (involve more keystrokes) than the HP-67 program because of the added prompting and other convenience features. But there are additional complications as well. For instance, recall that the HP-67 automatically displays both the step number and the keycode(s) following each program step that you key in. This is very handy because it enables you to verify that you in fact did press the key(s) you were supposed to press and not some other key(s) by mistake. The TI-59 does not do this automatically, although you can do it manually (less convenient). The HP-41C does not display the keycode after each program step either, but it does display an "alpha mnemonic" along with the step number. For instance, suppose the square-root operation were step number 275. After keying in this step, the HP-41C would immediately display "275 SQRT." This particular mnemonic may seem fairly obvious, but there are several others that would be quite foreign to someone who had not read the manual in some detail. Thus, to make the HP-41C easy to program for someone with no background in math and with no desire to learn the calculator language, it is necessary to list both the keycodes and the alpha mnemonics, side-by-side. At Lou Day's suggestion, I have decided to make these programs available, for the price of the postage, to those Guild members who wish to have one. Just look up Lou (Lucius Day) in the Guild Directory for help with the TI-59 calculator and program or contact me for help with the HP-67 or HP-41C and corresponding programs. I hate to tell you what these programs would be worth if Lou and I actually charged you for the time we spent developing them. I am sure Dr. Al Sanderson could say the same about the TI-59 program he developed to facilitate the Guild national tuning exam.

Let me emphasize that these calculator programs are sophisticated tools intended primarily for the experienced rebuilder. The intent here is not to substitute for experience and common sense, but to add to them. It is important to keep in mind that the formulas implicit in these programs are actually a simplified mathematical description of an enormously complex instrument. Even so, I believe the piano scale evaluation/modification approach outlined in this book represents, for the first time, a reasonably sound, scientific point of departure for both scaling and rescaling work for our membership. It is certainly a significant advance beyond the simplistic notions of "equal tension," Klepac charts and the like, which do not deal at all properly with the important acoustical factors involved in good scale design.

Now let us return to the two new calculator programs which Lou Day and I developed. For comparative purposes, we programmed the TI-59 and HP-41C so that these two programs would be similar to each other and to the HP-67 program discussed at length earlier in Appendix 3. Assuming that anyone seriously interested in this sort of thing has already read this discussion, the summary of the TI-59 and HP-41C programs in Table A3.3 should be self-explanatory. To illustrate the differences among the three calculator programs, let us look at routine *A*. Recall that the HP-67 requires that you key in numerical values for **N**, **A**, **a** and **b** (not in this order) and then press the keybutton labelled *A*. The *A* routine then stores these numbers in memory for future use and stops (about 1 second). That's it. The TI-59 program is a little different, as you can see. Here, you start by pressing the *A* keybutton first. Then the number "6" flashes briefly in the display, followed by a steady display of the current value of **N** in the calculator's memory. The advantages here are three-fold. First, the "6" is a reminder to you that you are going to be asked whether the **N** value (6th column on your worksheet, as Lou envisions it) in the calculator's memory is O.K. for the unison you are about to analyze or modify. Secondly, if you want to change this number, the old number shows you the number of significant digits to use when you key in the new number. Thirdly, if the old number is O.K., you do not have to key in any number at all, thus saving on wearisome keystrokes. Instead, you just press the R/S (RUN/STOP) key, as indicated in step (ii), and continue in similar fashion with **A**, **a** and **b**. Finally, the HP-41C is still different because it asks you outright "N = 1 STRINGS?," followed by a short, low-pitched tone. There is hardly any question what the calculator is asking you here. The question mark together with the tone makes double sure you realize this is a question. When routine *A* is complete, the HP-41C flashes the message "PRESS B OR C," instead of simply stopping with zeros in the display, as does the HP-67 and TI-59. In other words, you are being told that the *A* routine is finished and you must next proceed to the evaluation/modification routines *B* or *C*.

The remaining routines follow a similar pattern, as you can see in Table A3.3. The HP-41C will continue to show question marks and emit a low-pitched tone whenever it asks questions. It will omit the question mark and emit a high-pitched tone whenever a calculated number (with appropriate units, if there is room) appears in the display. In this case you write down the calculated number on your worksheet and the next number and the next, in the order that the calculator displays them in succession. You will note that the TI-59 program is a little different here in that you have to press the R/S key each time before the next calculated quantity appears in the display. Lou likes the idea of writing down the calculated quantities at the user's own pace, whereas my thought was to save button pushing and have the calculator pace your writing speed. The HP-41C could easily be programmed according to Lou's preference if one so wished.

In routine *D*, example calculated values of new wound string dimensions $L_1$ and $L_2$ appear in the stringmaker's language (not decimal fractions), one after the other, as "$L_1$ = 6-1/2," then "$L_2$ = 78-1/16," using the HP-41C. The fraction 8/16 has automatically been reduced to its lowest denominator, in this case 1/2, before being displayed. The TI-59 cannot display both integer and fractional parts simultaneously in this same way, so Lou has programmed it to display the integer and fractional parts separately, just as I did with the HP-67. This whole routine would have to be rewritten if the metric system were being used, but the other routines would only have to be modified slightly. Metric versions are now available from me for both the HP-67 and HP-41C calculators.

# Table A3.3. TI-59 and HP-41C Piano Scale Evaluation/Modification Routines

| Routine | step | User Instruction | Display, TI-59 | Display, HP-41C | Comments |
|---|---|---|---|---|---|
| A | (i) | press *A* key | "6", then previous value of **N** | N=1 STRINGS? | *Preliminary routine:*<br>If **A**=0, HP-41C skips directly to the message "PRESS B OR C" upon pressing the *R/S* key following step (ii); TI-59 clears display and stops routine. HP-41C has second display indicating status of current unison, i.e., plain, wound, trichord, bichord or monochord. |
| | (ii) | press *R/S* key | "7", then previous value of **A** | A=0.89? | |
| | (iii) | press *R/S* key | "8", then previous value of **a** | a=0.6 INCH.? | |
| | (iv) | press *R/S* key | "9", then previous value of **b** | b=0.6 INCH.? | |
| | (v) | press *R/S* key | clear display and stop | PRESS B OR C | |
| *B* | (i) | press *B* key | "1", then previous value of **m** | SMALL M=1? | *Main evaluation/modification routine:*<br>If string not wound (i.e., if **A**=0), then TI-59 and HP-41C both skip directly to calculation and display of $I_4$, upon pressing *R/S* key following step (iii); note that HP-41C is programmed to display $d_W$, $I_4$, **Z** and $T/T_B$ in automatic succession. With TI-59 program, user presses *R/S* key to display **Z** after $I_4$, $T/T_B$ after **Z** and **T** after **NT/H**. For simplicity, calculators use **H**=**L**/**8**, which is approximately correct (or should be) only at the bass/treble break. It is not a problem that this relationship changes throughout the scale, as long as it is done smoothly. |
| | (ii) | press *R/S* key | "2", then previous value of **L** | L=79.5 INCH.? | |
| | (iii) | press *R/S* key | "3", then previous value of **d** | d=67 MILS? | |
| | (iv) | press *R/S* key | "4", then previous value of **D** | D=187 MILS? | |
| | (v) | press *R/S* key | "5", then calculated value of $d_W$, then | dW=63.2 MILS | |
| | (vi) | | "10", then calculated value of $I_4$ | then I4=3.0 CENTS | |
| | (vii) | press *R/S* key (TI-59 only) | "11", then calculated value of **Z** | then Z=3318 | |
| | (viii) | press *R/S* key (TI-59 only) | "12", then calculated value of $T/T_B$ | then T/TB=0.34 | |
| | (ix) | press *R/S* key | "13", then calculated value of **NT/H** | NT/H=35 | |
| | (x) | press *R/S* key (TI-59 only) | "14", then calculated value of **T** | then T=348 LBS. | |
| *C* | | press *C* key | Identical to routine *B*, except steps (iv) and (v) are reversed, i.e., $d_W$ is specified and **D** is then calculated | | This routine is only for wound string design/modification, where $d_W$ is specified. |
| *A'* | | press *A'* key (*a* on HP-41C) | Repeats *C* routine, steps (iii) and (v) through (x) | | Repeats *C* routine, changing only **d**. |
| *B'* | | press *B'* key (*b* on HP-41C) | Repeats *C* routine, steps (iv) through (x) | | Repeats *C* routine, changing only $d_W$. |
| *C'* | | press *C'* key (*c* on HP-41C) | Repeats *C* routine, steps (iii) through (x) | | Repeats *C* routine, changing just **d** & $d_W$. |
| *D'* | | press *D'* key (*d* on HP-41C) | Repeats *A* routine, steps (iii) & (iv), then the *C* routine, steps (vi) through (x) | | Recalculates $I_4$, changing only **a** & **b**. |
| | | press *D* key<br>press *R/S*<br>press *R/S* (TI-59 only)<br>press *R/S* (TI-59 only)<br>press *R/S* (TI-59 only) | "15", then previous value of **M**<br>"16", then integer part of $L_1$<br>"17", then fractional part of $L_1$<br>"18", then integer part of $L_2$<br>"19", then fractional part of $L_2$ | M=?<br>L1=6½<br>then<br>L2=78 1/16 | Using HP-41C example numbers for $L_1$ & $L_2$, TI-59 would display, in succession, "16", then "6"<br>"17", then "8"<br>"18", then "78"<br>"19", then "1"<br>User would then have to reduce fractions to lowest denominator for the stringmaker. |
| | | press *E* key | calculated $E_L$ | E=0.27 INCH | If **G**=**L**, then routines *E* and *E'* would give the same answer for the calculated value of elongation **E**. |
| *E'* | | key-in any string segment **G**, then press *E* key (*e* on HP-41C) | calculate $E_G$ | E=0.31 INCH | |

The prompt numbers in the TI-59 program correspond roughly to the column locations in the Chapter 9 example worksheet.

# References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